Redhill
A-Level
Maths

Workbook
A2 Pure 2

ALGEBRA and FUNCTIONS

Reminder - Algebraic fractions:

Addition and subtraction of algebraic fractions.

Example:

$$\frac{1}{x+1}+\frac{2}{x+3}\equiv\frac{1(x+3)}{(x+1)(x+3)}+\frac{2(x+1)}{(x+3)(x+1)}$$

$$\equiv\frac{x+3+2x+2}{(x+1)(x+3)}$$

$$\equiv\frac{3x+5}{(x+1)(x+3)}$$

Notice this is an identity - you are trying to find an equivalent fraction so that both sides are identical and therefore true for all x. Often = signs are used.

You need a common denominator. Multiply top and bottom of the first fraction by (x+3). Multiply top and bottom of the second fraction by (x+1).

Write over the common denominator.

Tidy up (collect like terms).

Complete these:

1.

$$\frac{1}{x}+\frac{1}{x+1}\equiv\frac{\qquad}{x(x+1)}+\frac{\qquad}{x(x+1)}$$

You need a common denominator. Multiply top and bottom of the first fraction by (x+1). Multiply top and bottom of the second fraction by x.

$$\equiv\frac{\qquad\qquad}{x(x+1)}$$

Write over the common denominator.

$$\equiv\frac{\qquad\qquad}{x(x+1)}$$

Tidy up (collect like terms).

2.

$$\frac{1}{x(x+1)}-\frac{3}{(x+1)(x+2)}\equiv\frac{\qquad}{x(x+1)(x+2)}-\frac{\qquad}{x(x+1)(x+2)}$$

You need a common denominator. Look for the different factors in both denominators. Multiply top and bottom of the first fraction by (x+2). Multiply top and bottom of the second fraction by x.

$$\equiv\frac{\qquad\qquad}{x(x+1)(x+2)}$$

Write over the common denominator.

$$\equiv\frac{\qquad\qquad}{x(x+1)(x+2)}$$

Tidy up (collect like terms).

3.

$$\frac{2x}{(x+2)^2}+\frac{3}{x+2}+\frac{1}{x}\equiv\frac{\qquad}{x(x+2)^2}+\frac{\qquad}{x(x+2)^2}+\frac{\qquad}{x(x+2)^2}$$

You need a common denominator. Look for the different factors in all denominators. Including repeated factors. Multiply top and bottom of the first fraction by x. Multiply top and bottom of the second fraction by x(x+2). Multiply top and bottom of the third fraction by (x+2) ^{2}.

$$\equiv\frac{\qquad\qquad}{x(x+2)^2}$$

Write over the common denominator.

$$\equiv\frac{\qquad\qquad}{x(x+2)^2}$$

Tidy up (collect like terms).

4.

$$2+\frac{5}{x-4}\equiv\frac{\qquad}{x-4}+\frac{\qquad}{x-4}$$

Remember 2 = $\frac{2}{1}$.

You need a common denominator. Multiply top and bottom of the first fraction by (x – 4)

$$\equiv\frac{\qquad\qquad}{x(x-4)}$$

Write over the common denominator.

$$\equiv\frac{\qquad\qquad}{x(x-4)}$$

Tidy up (collect like terms).

Reminder - factor theorem and algebraic division:

If $(x-a)$ is a factor of $f(x)$ then $f(a)=0$
If $f(a)=0$ then $(x-a)$ is a factor of $f(x)$

and

If $(ax-b)$ is a factor of $f(x)$ then $f(^{b}/_{a})=0$
If $f(^{b}/_{a})=0$ then $(ax-b)$ is a factor of $f(x)$

5. Determine whether or not $(x+3)$ is a factor of $x^3-11x-6$

6. If $(x+1)$ is a factor of x^3-ax+6 find a

7. Find the values a and b if $(x+2)$ and $(2x+1)$ are both factors of $2x^3+ax^2+bx-2$

8. Factorise $4x^3+4x^2-x-1$

Algebraic division:
Example: Divide $4x^3-4x^2-21x-9$ by $(2x+3)$

2x into $4x^3$ or what times $2x$ equals $4x^3$?
Answer: $2x^2$

$$\begin{array}{r} 2x^2 \qquad\qquad\quad \\ 2x+3\,\overline{)\,4x^3-4x^2-21x-9} \\ -(4x^3+6x^2 \qquad\qquad\;) \\ \hline -10x^2-21x-9 \end{array}$$

$2x^2$ times $2x+3$ and subtract

2x into $-4x^2$ or what times $2x$ equals $-4x^2$?
Answer: $-5x$

$$\begin{array}{r} 2x^2-5x \qquad\quad \\ 2x+3\,\overline{)\,4x^3-4x^2-21x-9} \\ -(4x^3+6x^2 \qquad\qquad\;) \\ \hline -10x^2-21x-9 \\ -(-10x^2-15x \qquad) \\ \hline -6x-9 \end{array}$$

$-5x$ times $2x+3$ and subtract

2x into $-6x$ or what times $2x$ equals $-6x$?
Answer: -3

$$\begin{array}{r} 2x^2-5x-3 \\ 2x+3\,\overline{)\,4x^3-4x^2-21x-9} \\ -(4x^3+6x^2 \qquad\qquad\;) \\ \hline -10x^2-21x-9 \\ -(-10x^2-15x \qquad) \\ \hline -6x-9 \\ -(-6x-9) \\ \hline ---- \end{array}$$

-3 times $2x+3$ and subtract

Remainder

Note:
Quotient is $2x^2-5x-3$
Remainder is 0

$\therefore\ 4x^3-4x^2-21x-9=(2x+3)(2x^2-5x-3)=(x+3)(2x+1)(x-3)$

Alternatively you can use inspection or by equating coefficients.

Improper fractions:

The degree of a polynomial is the highest power of x in the expression. i.e. $x^2 + 3x$ is of degree 2. $x^4 - x^2 + 5$ is of degree 4, $(x+4)(x-3)(x-5)$ is of degree 3 (when expanded the highest power of x is 3). If the numerator has degree equal to, or greater than, the denominator then the expression can be rewritten as shown below and the above method applied (or you can use algebraic division).

$$\frac{x+1}{x+2} \equiv A + \frac{B}{x+2}$$

$\frac{x^2+2}{x+2} \equiv Ax + B + \frac{C}{x+2}$ and so on Note: here you could also divide and put the remainder over $x + 2$

$\frac{2x^2}{2x^2+5x-3} \equiv A + \frac{B}{2x^2+5x-3}$ or factorise the denominator and so $\frac{2x^2}{(x+3)(2x-1)} \equiv A + \frac{B}{(x+3)} + \frac{C}{(2x-1)}$

$$\frac{x^3}{(x+3)(2x-1)} \equiv Ax + B + \frac{C}{(x+3)} + \frac{D}{(2x-1)}$$

and so on.

Example: Express $\frac{x+1}{x+2}$ in the form $A + \frac{B}{x+2}$

$$\frac{x+1}{x+2} \equiv A + \frac{B}{x+2}$$

$$\equiv \frac{A(x+2)}{x+2} + \frac{B}{x+2} = \frac{A(x+2)+B}{x+2}$$

So $x + 1 \equiv A(x+2) + B$

This is true for all x.

$x = -2$ $-1 = B$

$x = 0$ $1 = 2A + B = 2A - 1$

$2A = 2 \Rightarrow A = 1$

So $\frac{x+1}{x+2} \equiv 1 - \frac{1}{x+2}$

You could use division but it is useful to know this method.

As before you need a common denominator. Think of A as $\frac{A}{1}$ *Multiply top and bottom of the fraction by (x + 2) and write over the common denominator.*

True for all x, so you can substitute any values of x to enable you to find A and B. Easiest to choose values that make the bracket zero. Any values will do but you will have to solve simultaneous equations if you do not use value(s) that make the bracket(s) zero.
No other brackets so choose a simple value for x.

Rewrite the original expression.

9. Express $\frac{4x^2+3}{2x+1}$ in the form $Ax + B + \frac{C}{2x+1}$

Ax is $\frac{Ax}{1}$ *and B is* $\frac{B}{1}$

Write over the common denominator.

True for all x.

Rewrite the original fraction in partial fractions.

OR you can do this question by algebraic division.

10. Express $\frac{2x-3}{x+2}$ in the form $A + \frac{B}{x+2}$

11. Express $\frac{2x^2+2}{(x+1)(x-3)}$ in the form $A+\frac{B}{(x+1)}+\frac{C}{(x-3)}$

Think of A as $\frac{A}{1}$ Multiply top and bottom of the fraction by (x + 1)(x – 3). Multiply top and bottom of the second fraction by (x – 3) and the third by (x + 1). Write over the common denominator.

True for all x, so you can substitute any values of x to enable you to find A, B and C.

Easiest to choose values that make the brackets zero.
You should be able to find two. Then choose a simple value for x.
x = 0 perhaps.

Rewrite the original fraction in partial fractions.

Simplifying rational expressions:

With fractions always look to factorise and/or cancel.

12. If $f(x) = 3x^3 + 5x^2 - 4x - 4$

(a) Find f(1)

(b) Use the factor theorem to show that $(3x + 2)$ is a factor of $f(x)$

(c) Simplify $\frac{3x^3+5x^2-4x-4}{3x^2-x-2}$

13. Simplify (a) $\frac{\frac{1}{t}-1}{1-t}$ (b) $\frac{x^2+2x}{\frac{x}{2}+1}$

14. Simplify $\frac{2x(2x+3)(2x-1)+(2x+3)(2x-9)}{4x^2+12x+9}$

Partial Fractions:

Learn the methods for the different types.

15. **Type I Denominators of the form $(ax+b)(cx+d)$** Express $\frac{2x-3}{(x+1)(x-2)}$ as the sum of partial fractions. $\frac{2x-3}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$	*Put a constant over each linear factor and add. Notice this is an identity – you are trying to find A and B so that both sides are identical and therefore true for all x.*
$\equiv \frac{\qquad}{(x+1)(x-2)} + \frac{\qquad}{(x+1)(x-2)}$	*As before you need a common denominator. Multiply top and bottom of the first fraction by (x–2). Multiply top and bottom of the second fraction by (x+1). You may miss out this line.*
$\equiv \frac{\qquad}{(x+1)(x-2)}$	*Write over the common denominator. Best to leave with the brackets.*
So $2x-3 \equiv A(x-2)+B(x+1)$ **This is true for all *x*** $x=2 \quad 1 = 3B \quad \Rightarrow \quad B = ⅓$ $x = \quad = \quad \Rightarrow \quad A =$	*True for all x, so you can substitute any values of x to enable you to find A and B. Easiest to choose values that make the bracket zero. (Any values will do but you will have to solve simultaneous equations if you do not use values that make the brackets zero.)*
$\frac{2x-3}{(x+1)(x-2)} \equiv \frac{\qquad}{(x+1)} + \frac{\qquad}{(x-2)}$	*Rewrite the original fraction in partial fractions. Remember your fraction work.* $\frac{\frac{1}{3}}{x-2} = \frac{1}{3(x-2)}$

16. Write $\frac{2x+1}{(x-2)(x+4)}$ as the sum of partial fractions.

17. Express $\frac{4-5x}{(2x-1)(x-1)}$ as the sum of partial fractions.

18. **Type II Denominators of the form $(ax + b)(cx + d)^2$**

Express $\frac{2x}{(x+3)(x-1)^2}$ as the sum of partial fractions.

$$\frac{2x}{(x+3)(x-1)^2} \equiv \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Notice for the squared bracket you write as two partial fractions, one without a square and one with a square – and constant on the top of both.

$$\equiv \frac{}{(x+3)(x-1)^2} + \frac{}{(x+3)(x-1)^2} + \frac{}{(x+3)(x-1)^2}$$

Add the fractions. (You can miss out this line.)

$$\equiv \frac{}{(x+3)(x-1)^2}$$

So $2x \equiv$ A( ) + B( )( ) + C( )

This is true for all x.

$x =$

$x =$

$x =$

True for all x, so you can substitute any values of x to enable you to find A, B and C.
Easiest to choose values that make the brackets zero.
You should be able to find two. Then choose a simple value for x. x=0 perhaps.

$$\frac{2x}{(x+3)(x-1)^2} \equiv \frac{}{(x+3)} + \frac{}{(x-1)} + \frac{}{(x-1)^2}$$

Rewrite the original fraction in partial fractions.

19. Decompose $\frac{x+1}{(x-1)(x+2)^2}$ into partial fractions.

20. Express $\frac{3}{x^2(x+3)}$ in partial fractions.

Careful. You should have three fractions.

21. Write $\frac{4x}{(x-3)(2x+1)}$ as the sum of partial fractions.

22. Express $\frac{x^2+2}{(x+2)(x^2-1)}$ in partial fractions.	*Any quadratic factors should factorise.* *e.g.* $x^2-4=(x+2)(x-2)$ *Here you should spot 'the difference of two squares' and factorise.* *Then as before but with three partial fractions.*

Partial fractions and the binomial expansion:

23. Express $\frac{x}{(x+2)(x+1)}$ in partial fractions and hence express $\frac{x}{(x+2)(x+1)}$ as a series in ascending powers of x up to the term in x^3 State the range of values for which the expansion is valid.	*Express in partial fractions and then rewrite each denominator on the top of the fraction in brackets to a negative power.* *Do not forget: you need* $(1+\quad)^{power}$ *to use the expansion.* *(x+2) = (2+x) so take 2 out as a factor before doing the expansion.* *Expand each term and collect together like terms. Careful with the minus.* *x has to be valid for both expansions so pick the one with the tighter restrictions.*

Partial fractions and differentiation:

24. Express $f(x) = \dfrac{2x}{(2x+1)(x-2)}$ in partial fractions and hence differentiate $f(x)$

Express in partial fractions and then rewrite each denominator on the top of the fraction in brackets to a negative power.
Differentiate remembering:
If $y = (ax + b)^n$
then $\dfrac{dy}{dx} = an(ax + b)^{n-1}$

Partial fractions and integration - see page 28.

Mixed questions 1

25. The polynomial $f(x)$ is defined by $f(x) = 2x^3 - 3x^2 - 11x + 6$
(a) Find $f(-2)$
(b) Show that $(2x - 1)$ is a factor of $f(x)$
(c) Factorise $f(x)$ as far as possible.

26. Solve the equation $3x^3 + 4x^2 - 3x - 4 = 0$

27. Express $\frac{9x^2+1}{3x-1}$ in the form $Ax + B + \frac{C}{3x-1}$

28. (a) Show that $(x + 1)$ is a factor of $3x^3 - 2x^2 - 3x + 2$

(b) Simplify $\dfrac{3x^3 - 2x^2 - 3x + 2}{6x^2 - x - 2}$

29. Express as a single fraction $\frac{3}{1-2x} - \frac{1}{2+x}$

30. Write $\frac{2-3x}{(1-2x)(1-x)(2+x)}$ as the sum of partial fractions.

31. $f(x) = \frac{5}{x(x+1)^2}$ (a) Express $f(x)$ as the sum of partial fractions. (b) Find $f'(x)$

32. $f(x) = \frac{5x+1}{(2x+1)(x-1)} = \frac{1}{(2x+1)} + \frac{2}{(x-1)}$ Express $f(x)$ as a series in ascending powers of x up to the term in x^3 State the range of values for which the expansion is valid.

DIFFERENTIATION

Rates of change:

$\frac{dy}{dx}$ is the rate at which y changes as x changes.

$\frac{dy}{dx} = 3$ means y is changing 3 times as fast as x (If x changes by 5, y will change by 15).

The rate of change of p with z is $\frac{dp}{dz}$

The rate of change of q is $\frac{dq}{dt}$ (always assumed with time if not given).

If $\frac{dq}{dt}$ is + ve then q is increasing.

If $\frac{dq}{dt}$ is – ve then q is decreasing.

> You may also need to use the fact
> $$\frac{dy}{dx} = \frac{1}{{}^{dx}/_{dy}}$$

For rate of change problems write down:

- Any formula that may be used (area, volume,…) - you will probably need to differentiate this.
- Any rate of change given. You may be given a rate of change directly proportional to a variable - so you need to know how to write this as an equation with a constant of proportionality.
- The rate of change that you have to find.

You use the 'chain rule' to find the rate of change required.

33. Find the rate of increase of the volume of a cube when the side is 3.5 cm and each side length is increasing at a rate of 0.3 cm^2/sec. Complete: Find – given $\frac{dx}{dt} =$ $V =$ $\frac{dV}{dx} =$ $\frac{dV}{dt} = \frac{dV}{\quad} \times \frac{\quad}{dt} =$ $x = 3.5 \quad \frac{dV}{dt} = \qquad =$	*Let side length of the cube be x.* *Volume of a cube.* *Differentiate.* *Use the chain rule.*
34. On a hot day a circular puddle of water is decreasing in area at a constant rate of 20 cm^2 s^{-1}. Find the rate of decrease of the radius of the puddle when its area is 1 m^2.	*Note: Area 1m^2= 10000cm^2.* $\frac{dA}{dt}$ *is negative (it is decreasing)* *Want to find* $\frac{dr}{dt}$ *when* $A = 10000cm^2$. *Use* $A=$ $\qquad \frac{dA}{dr} =$ $\frac{dr}{dt}$ *using the chain rule.* *Find r when A = 10000 and hence* $\frac{dr}{dt}$
35. The surface area of a sphere is increasing at a rate of 3 cm^2 s^{-1}. Find the rate of increase, in cm^3 s^{-1}, of the volume of the sphere when its radius is 4 cm.	*Want* $\frac{dV}{dt}$ *when* $r = 4$ $V=\frac{4}{3}\pi r^3$ *hence* $\frac{dV}{dr}$ $S=4\pi r^2$ *hence* $\frac{ds}{dr}$ *Given* $\frac{ds}{dt} = 3$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ *but do not know* $\frac{dr}{dt}$ *Find* $\frac{dr}{dt}$ *Find* $\frac{dV}{dt}$ *when r=4.*

You may be given a rate of change directly proportional to a variable - so you need to know how to write this as an equation with a constant of proportionality. You could be given further information to find this constant.

36. During initial growth the cells in a culture increases in proportion to the number of cells present. Write down a differential equation to model the rate at which the number of cells, *n*, is increasing with time, *t*.	*The number of cells, n, increases with time so dn/dt.*
37. If a body falls, from rest, through a medium which causes the velocity to decrease at a rate proportional to the velocity, write down a differential equation that satisfies this motion.	*Velocity decreases with time so – dv/dt.*

Also see page 33.

Implicit differentiation:

For some equations it is impossible to rearrange to give $y = f(x)$ and hence differentiate. So you use 'the chain rule' to differentiate each term in an equation without rearrangement. You can use this method (and it is often convenient and simpler to do so) for any equation. Below is the method you should use to differentiate terms (with y in them) with respect to x. You may do without the initial writing down of 'the chain rule' but you are advised to start by using it and leave it out later.

Examples (i) $\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$	*(i) To differentiate y^2 with respect to x just differentiate with respect to y (i.e. 2y) and times by $\frac{dy}{dx}$*
(ii) $\frac{d(y)}{dx} = \frac{dy}{dx}$	*(ii) The differential of y with respect to x is $\frac{dy}{dx}$*
(iii) $\frac{d(y^3)}{dx} = \frac{d(y^3)}{dy} \times \frac{dy}{dx} = 3y^2 \times \frac{dy}{dx}$	*(iii) To differentiate y^3 with respect to x just differentiate with respect to y (i.e. $3y^2$) and times by $\frac{dy}{dx}$*
(iv) $\frac{d(2y)}{dx} = \frac{d(2y)}{dy} \times \frac{dy}{dx} = 2 \times \frac{dy}{dx}$	*(iv) To differentiate 2y with respect to x just differentiate with respect to y (i.e. 2) and times by $\frac{dy}{dx}$*
(v) $\frac{d(xy)}{dx} = x\frac{dy}{dx} + y \times 1 = x\frac{dy}{dx} + y$	*(v) and (vi) For these two you are differentiating a product. The differential of xy - This is x times f(x) (y = f(x) is implied). The first function times the differential of the second function plus the second function times the differential of the firs function t. When you differentiate the y term use the methods above.*
(vi) $\frac{d(x^2y^2)}{dx} = x^2\frac{d(y^2)}{dy} \times \frac{dy}{dx} + y^2 \times 2x = x^2 2y\frac{dy}{dx} + 2xy^2$	
(vii) $\frac{d(\sin y)}{dx} = \frac{d(\sin y)}{dy} \times \frac{dy}{dx} = \cos y \frac{dy}{dx}$	*(vii) To differentiate sin y with respect to x just differentiate with respect to y (i.e. cosy) and times by $\frac{dy}{dx}$*
(viii) $\frac{d(e^y)}{dx} = \frac{d(e^y)}{dy} \times \frac{dy}{dx} = e^y \frac{dy}{dx}$	*(viii) To differentiate e^y with respect to x just differentiate with respect to y (i.e. e^y) and times by $\frac{dy}{dx}$*
Complete:	
38. $\frac{d(y^4)}{dx} =$	*38. To differentiate y^4 with respect to x just differentiate with respect to y and times by $\frac{dy}{dx}$*
39. $\frac{d(\ln y)}{dx} =$	
40. $\frac{d(e^{-y})}{dx} =$	*40. Careful here. e^{-y} is the same as e^{-1y}*

Complete:	
41. $\frac{d(5xy^3)}{dx} =$	*41. A product of 5x and* y^3
42. $\frac{d(3\cos y)}{dx} =$	
43. $\frac{d(x\sin 2y)}{dx} =$	*43. A product of x and sin 2y.*
44. $\frac{d(x^2 \tan y)}{dx} =$	*44. A product.*
45. $\frac{d(x\ln y)}{dx} =$	*45. A product.*
Differentiate each equation with respect to x and hence find $\frac{dy}{dx}$	*You can differentiate each term separately and then combine them or differentiate each term in the equation (but be careful!)*
46. $x^2+y^2=25$ $\frac{d(x^2)}{dx} =$ $\frac{d(y^2)}{dx} =$ $\frac{d(25)}{dx} =$	*46. Here differentiate each term separately.*
$=$	*Combine them.*
$=$	*All terms with* $\frac{dy}{dx}$ *in on one side all other terms on the other.*
$\frac{dy}{dx} =$	*Now rearrange to get* $\frac{dy}{dx} =$
47. $2x^2+xy-y=2$ $\frac{d(2x^2)}{dx} =$ $\frac{d(xy)}{dx} =$ $\frac{d(y)}{dx} =$ $\frac{d(2)}{dx} =$	*47. Again differentiate each term separately.*
$=$	*Combine them. Careful - there is a minus.*
$=$	*All terms with* $\frac{dy}{dx}$ *in on one side all other terms on the other.*
$\frac{dy}{dx}(\qquad\qquad) =$	*Take* $\frac{dy}{dx}$ *out as a factor.*
$\frac{dy}{dx} =$	*Now rearrange to get* $\frac{dy}{dx} =$
48. $y=x^2 e^y$	*48. Careful - a product and it is e to the power y.* *All terms with* $\frac{dy}{dx}$ *in on one side all other terms on the other.* *Take* $\frac{dy}{dx}$ *out as a factor.* *Now rearrange to get* $\frac{dy}{dx} =$
49. $\frac{1}{x}+\frac{1}{y}=\ln y$	*49. First write the reciprocals as negative powers. Then differentiate each term.* *All terms with* $\frac{dy}{dx}$ *in on one side all other terms on the other.* *Take* $\frac{dy}{dx}$ *out as a factor.* *Now rearrange to get* $\frac{dy}{dx} =$ *and tidy up.* *Note: You do not want fractions within fractions.*

50. Find the equation of the tangent to $x^2 - 3y^2 = 4y$ at the point $(8, 4)$

51. If $y^2 = 4x + 1$, find $\frac{dy}{dx}$ in terms of x.

52. Find the equation of the tangent to the circle $x^2 + y^2 + 4x + 6y = 0$ at the point $(1, -1)$

Differentiating a^x :

Let $y = a^x$

Taking logs base e $\ln y = \ln a^x$

Using log laws $\ln y = x \ln a$

Differentiate $\frac{1}{y}\frac{dy}{dx} = \ln a$

$$\frac{dy}{dx} = y \ln a = a^x \ln a$$

Learn this method as you may be asked to show how the result is obtained.

So $\boxed{\dfrac{d(a^x)}{dx} = a^x \ln a}$

Example: $\dfrac{d(7^x)}{dx} = 7^x \ln 7$	*Just use the rule.*
53. Find $\frac{dy}{dx}$ if $y = 4^{3x}$ Let $u = 3x$ $\frac{du}{dx} =$ So $y = 4^u$ and $\frac{dy}{du} =$ $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} =$	*Using substitution.* *Use the rule for* $\frac{dy}{du}$ Now use the chain rule.

54. Differentiate

(a) 2^x (b) 3^{2x+1}

(c) 5^{x^2} (d) 6^{-x}

(a) Just use the rule.

(b) Chain rule. Write down the answer or use substitution.

(c) Chain rule.

(d) Careful. Chain rule.

Reminder: Exponential functions are of the form a^x e.g. 2^x , 3^x , 10^x , etc. *The exponential function* is e^x

$$\frac{d(a^x)}{dx} = a^x \ln a \qquad \frac{d(e^x)}{dx} = e^x \qquad \frac{d(e^{ax})}{dx} = ae^{ax}$$

55. The number of bacteria, *n*, at time *t* minutes is given by $n = \mathrm{A}\,2^t$, where A is a constant. Initially there are 5 bacteria.

(i) Find the number of bacteria after 10 minutes.

(ii) Find the rate at which the bacteria is growing after 10 minutes.

*Initially means when $t = 0$
First use this information to find A.
Then you can find n when $t = 10$.*

Rate of growth (with time) so differentiate.

56. The mass of a radioactive substance is given by $m = m_0\, e^{-kt}$ where m is the mass at time *t* years, m_0 is the initial mass and *k* is a constant.
Given that $m = 0.8\, m_0$ when $t = 6$ find the mass in terms of m_0 when $t = 10$.
Also find the time when $m = 0.5\, m_0$

57. The population of a town can be modelled by $\mathrm{P} = 26500 \times 1.04^t$ where *t* is the number of years from 2018. (a) What is the population in 2018?

(b) What is the rate of growth of the population after 5 years?

58. You buy a car for £8000 and it depreciates by 20% per year. Write down an exponential equation to model this depreciation. How much will the car be worth after 5 years?

Describing curves:

Join two points on the curve in the region you are looking at
- if above the curve then that part of the curve is described as '**convex**' or '**concave upwards**' 
- if below the curve then that part of the curve is described as '**concave**' or '**concave downwards**' 

Turning points are where $\frac{dy}{dx}=0$ and the gradient changes from negative to positive as x increases ($\frac{d^2y}{dx^2}$ is positive) for a minimum, or from positive to negative as x increases ($\frac{d^2y}{dx^2}$ is negative) for a maximum.

Stationary points are where $\frac{dy}{dx}=0$ These could be local maximums ($\frac{d^2y}{dx^2}$ is negative), local minimums ($\frac{d^2y}{dx^2}$ is positive) or stationary points of inflection ($\frac{dy}{dx}$ does not change sign as the curve passes through the point).

Stationary points of inflection are where $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=0$ **and** the curve changes from convex (concave upwards) to concave (concave downwards) or vice versa as the curve passes through the point. i.e. $\frac{d^2y}{dx^2}$ changes sign. $\frac{dy}{dx}$ does not change sign.

Non-stationary points of inflection are where $\frac{dy}{dx}\neq 0$, $\frac{d^2y}{dx^2}=0$ **and** the curve changes from convex (concave upwards) to concave (concave downwards) or vice versa as the curve passes through the point.
i.e. $\frac{d^2y}{dx^2}$ changes sign. $\frac{dy}{dx}$ does not change sign.

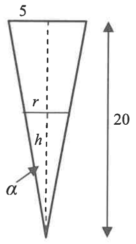
59. Find the point of inflection for the curve $y=x^3+3x^2-x-4$ and show that it is a non-stationary point of inflection.

60. Show that the curve $y=x\mathrm{e}^x$ has one point of inflection and give the coordinates of this point.

61. A cubic curve, $y = f(x)$, has a point of inflection at $(2, -30)$. The equation of the tangent of the curve at the origin is $y + 7x = 0$. Determine the equation of the curve.

Mixed questions 2

62. A container in the shape of an inverted right circular cone of height 20 cm and base radius 5 cm is being used to catch water dripping from a tap at a constant rate of 0.2 cm^3 per minute. If h is the depth of water in the cone, find the rate of change of h in cm per minute when the depth of water in the cone is half way up the cone. Give your answer to 2 decimal places.



63. Differentiate the following with respect to x

(a) x (b) y (c) x^3 (d) y^3 (e) $5xy^3$ (f) $3x^3y$

64. Differentiate the following with respect to x

(a) $2\cos 3x$ (b) $2\cos 3y$ (c) $2e^{3x}$ (d) $2e^{3y}$ (e) $2\ln 3y$

65. Find the equation of the tangent to the ellipse $x^2 + 2y^2 = 33$ at the point $(5, 2)$

66. Find $\frac{dy}{dx}$ if $y = 2^{3x}$

67. Find $\frac{dy}{dx}$ for $5x^2 - 3xy - 6y^2 = 8$

68. The number of rodents on an island is modelled by $N = A\,e^{kt}$ where N is the number of rodents after time, t months. Initially there were 6 rodents.

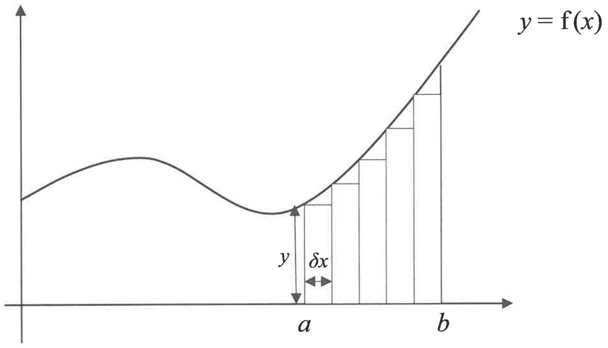
(a) What is the value of A?
(b) If there are 24 rodents after 2 months, work out the value of k.
(c) What is the population after 1 year?
(d) What is the rate of growth of the population after 6 months?
(e) Criticise this model.

69. Find the point of inflection for the curve $x^3 + 9x^2 + 15x - 4$ and show that it is a non-stationary point of inflection.

INTEGRATION

Integration as a limit of a sum:

Consider the area between the curve $y = \text{f}(x)$ the x axis and the lines $x = a$ and $x = b$
This area can be estimated by splitting the area into thin vertical rectangles.



The sum of the areas of the rectangles then gives an approximate value for the required area.
The thinner the strip, (and so there are more of them) the better the approximation.
The limit of this sum as the rectangles become narrower is the integral.

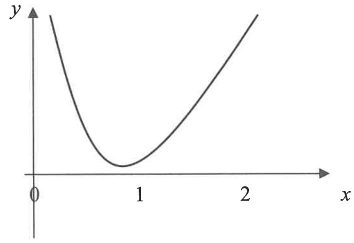
The fundamental theorem of calculus says that this is the same as doing the reverse of differentiation.

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx \qquad \therefore \textbf{ integration is a process of summation.}$$

Although you have applied this to area the result applies to any situation where you end up with a limit of a sum.

70. Part of the graph $y = \frac{1}{x^2} + x^3 + 1$ is shown. The area under the graph between the lines $x = 1$, $x = 2$ and the x axis is estimated by calculating the areas of the 5 rectangles, of equal width, δx.

(a) What is the value of δx ?



(b) Work an estimate for area under the graph using 5 rectangles, $\sum_{x=1}^{x=2} y\,\delta x$, where y is the height of the bars.

(c) Calculate $\lim_{\delta x \to 0} \sum_{x=1}^{x=2} y\,\delta x$

Reminder - integration:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad \int ax^n\ dx = a\frac{x^{n+1}}{n+1} + c$$

True for all rational n except for n = – 1

$$\int (ax+b)^n\ dx = \frac{(ax+b)^{n+1}}{a \times (n+1)} + c \qquad n \neq -1$$

A useful one to know - only works if the expression in the bracket is **linear**. (Differentiate the answer using the 'chain rule' to check its validity.)

Work out:

71. $\int x^3 + 3\sqrt{x} - 5\ dx$	72. $\int \frac{2}{x^2} - \frac{3}{x^3}\ dx$
73. $\int (3x-2)^4\ dx$	74. $\int 2(2x+1)^5\ dx$

75. $\int \frac{2}{(2x+1)^3}\, dx$ *Note: Write the bracket on the top with a negative power then you can use the above result.* *The numerator must be a constant, the expression in the bracket linear, and the power ≠ –1*	76. $\int \frac{2}{5(3x-1)^2}\, dx$

Area between two curves:

Reminder:

The area between a curve $y = \mathrm{f}(x)$ and the x axis for $x = a$ to $x = b$ is given by:

$$\text{Area between curve and } x \text{ axis} = \int_a^b y\, dx = \int_a^b f(x)\, dx$$

use the function of x in the integral.

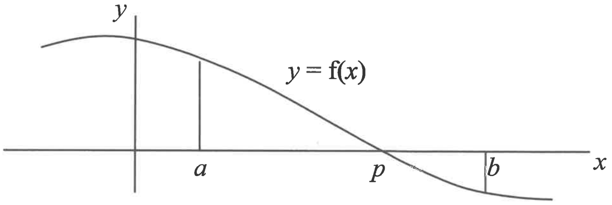
Areas below the x axis are negative.

Sketch the curve if possible.

For area between the curve $y = \mathrm{f}(x)$ and the x axis for $x = a$ to $x = b$ as shown below you will need to work out:

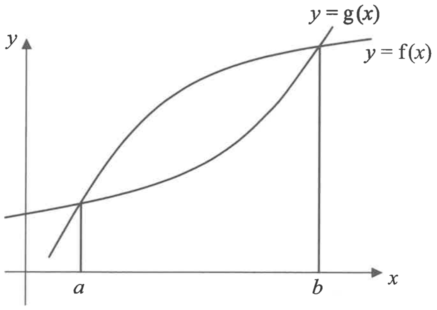
$$\int_a^p f(x)\,dx \quad \text{and} \quad \int_p^b f(x)\,dx$$

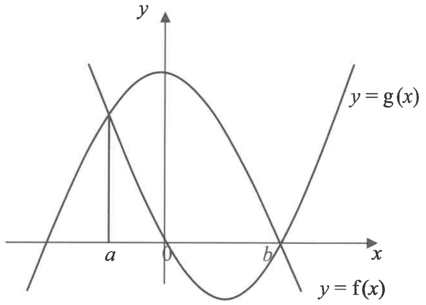
Make both areas positive and add them together.



You could be asked to work out the integral $\int_a^b f(x)\, dx$ If so, then do not worry about signs, just work it out!

Area between two curves:





Here it is easy to see that the area between the two curves equals the area under $y = \mathrm{f}(x)$ minus the area under $y = \mathrm{g}(x)$ from $x = a$ to $x = b$

Here the area between the curves equals the area under $y = \mathrm{f}(x)$ minus the area under $y = \mathrm{g}(x)$ from $x = a$ to $x = 0$. For the next part from $x = 0$ to $x = b$ it will be the same as the $\mathrm{g}(x)$ area will be negative and when you subtract it from the $\mathrm{f}(x)$ area this will make it a positive area to add.

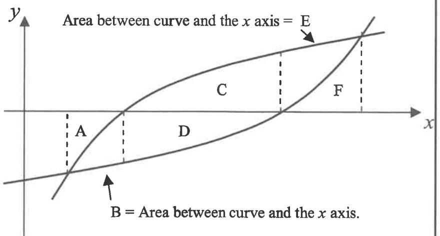
So for both graphs:

Area between the curves from $x = a$ to $x = b$ $= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

$$= \int_a^b f(x) - g(x)\, dx$$

So you do not need to worry about the parts of either curve above or below the x axis. And if you do the subtraction the other way round the area just works out to be negative.
All you need is the points of intersection.

Check yourself with these curves.



Just work out which areas will be positive or negative and what happens when you subtract the functions.

77. Find the area between the curves $y = x^2 - 3x - 18$ and $y = 9 - x^2$

A sketch will help you to 'see' what you are doing.

Find the points of intersection.

Find the area between the two curves.

MEI students: For areas between a curve and the **y axis** for $y = a$ to $y = b$ use:

Area between curve and y axis $= \int_a^b x\, dy = \int_a^b f(y)\, dy$

Rearrange (if necessary) to give x in terms of y and use the above.

78. MEI students: Find the area enclosed between the y axis and the curve $x = 3y - y^2$

A sketch will help you to 'see' what you are doing.

Integrating e^x and $\frac{1}{x}$:

Other standard results you should know:

$$\int e^x dx = e^x + c$$

$$\int ae^x dx = ae^x + c$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int \frac{1}{kx} dx = \frac{1}{k}\ln|x| + c$$

$$\int \frac{a}{kx} dx = \frac{a}{k}\ln|x| + c$$

$$\int \frac{a}{kx+b} dx = \frac{a}{k}\ln|kx+b| + c$$

Work out: 79. $\int 3e^x dx$ 80. $\int \frac{1}{2x} dx$ 81. $\int \frac{5}{7x} dx$ 82. $\int \frac{1}{3} e^x dx$ 83. $\int 5x^{-1} dx$ 84. $\int 5 - \frac{2}{3x} dx$ 85. $\int 5(e^x + 3)\ dx$ 86. $\int (1 - \frac{1}{x})^2 dx$ 87. $\int \frac{3}{x+2} dx$ 88. $\int \frac{1}{2x+3} dx$ 89. $\int \frac{3}{5x+2} dx$ 90. $\int \frac{4}{5-2x} dx$	*Use the rules above.* *83. Write as* $\int \frac{5}{x} dx$ *(only the x is to the power −1)* *84. Integrate each term.* *85. Expand the bracket. Then integrate.* *86. Expand the brackets. Write as two brackets - you are less likely to make a mistake. Remember ln x is the result for integrating 1/x . 1 over other powers of x in the denominator you must use indices and the power rule.*
91. Work out $\int e^{x} - \sqrt{x}\ dx$	*First rewrite the* $\sqrt{x}$ *as a fractional power.* *Integrate using the rules.*
92. Work out $\int \frac{4}{5x} + \frac{3}{2x^2}\ dx$	*The first term is* $\frac{4}{5} \times \frac{1}{x}$ *The second term is **not** a ln. Write as a power of x. This means put the x to the top as a negative power.*
93. Evaluate $\int_{-1}^{1} 2e^x + 2\ dx$	*Integrate each term and then substitute in the values of x. Top one substituted in minus the bottom one substituted in.*
94. Evaluate $\int_{1}^{2} \frac{3}{2x} dx$	*Integrate and then substitute in the values of x.* *Remember ln 1 = 0*
95. Find the area between the curve $y = x^2 + e^x$, the x axis and the lines $x = 1$ and $x = 2$	*A sketch of the curve helps to see what you are doing.*

Integrating $\sin x$ and $\cos x$:

$$\boxed{\int \sin x \, dx = -\cos x + c} \qquad \boxed{\int a \sin bx \, dx = \frac{-a\cos bx}{b} + c}$$

$$\boxed{\int \cos x \, dx = \sin x + c} \qquad \boxed{\int a \cos bx \, dx = \frac{a\sin bx}{b} + c}$$

and from the differentiation of $\tan x$ $\boxed{\int \sec^2 x \, dx = \tan x + c}$

96. $\int \sin 3x \, dx =$	97. $\int 2\cos 6x \, dx =$
98. $\int \sin\left(x - \frac{\pi}{4}\right) dx =$	99. $\int 3\cos(\pi - 2x)\, dx =$
100. $\int_0^{\pi/6} 2\sin 2x \, dx =$	101. $\int_{\pi/6}^{\pi/2} 3\cos\left(2x + \frac{\pi}{2}\right) dx =$
102. $\int 2\sec^2 x \, dx =$	103. $\int \sec^2 3x \, dx =$

Using Trigonometric identities:

104. $\int \sin^2 x \, dx$	*Use the identity* *$\cos 2x = 1 - 2\sin^2 x$.* *Rearrange to give $\sin^2 x$.* *Substitute and then integrate.*
105. $\int \tan x \, dx = \int \frac{\qquad}{\qquad} dx$	*Use the identity for $\tan x$.* *Then use* $\int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x)\rvert + c$ *Careful (mind the minus)* *OR* *use the substitution $u = \cos x$*
106. $\int 1 + \tan^2 x \, dx$	*Use the identity for $1 + \tan^2 x$.* *Then integrate. A standard result.*
107. $\int_0^{\pi/2} 4\cos^2 x \, dx$	*Use the identity for $\cos 2x$ in terms of $\cos x$.* *Rearrange to give $\cos^2 x$* *Then integrate and apply limits.*

108. Use the double angle identity to express $\cos 4\theta$ in terms of $\cos 2\theta$. Hence evaluate $\int_0^{\pi/4} \cos^2 2\theta \, d\theta$

Use the identity for $\cos 2x$ in terms of $\cos x$ with x replaced by θ
Rearrange to give $\cos^2 2\theta$.
Substitute and integrate.
Apply limits.

Integration by substitution:

(Change of variable)

You change the variable to put the integral into a form which you can integrate from the rules that you know. Learn the method - don't forget to change the *dx* .

Example: $\int x(x^2+3)^3 \, dx$

Let $u = x^2 + 3$ Then $\frac{du}{dx} = 2x$ *and so* $....\frac{1}{2}du = ...xdx$ $\left(\text{from } \int ...\frac{du}{dx}dx \equiv \int ... du \right)$

$$\therefore \int x(x^2+3)^3 dx = \int \tfrac{1}{2}u^3 du$$
$$= \frac{u^4}{2 \times 4}$$
$$= \tfrac{1}{8}(x^2+3)^4 + c$$

Look at the question if u is to be (x^2+3) then you need to replace $x\,dx$

109. $\int x(x^2+5)^5 \, dx$

Let $u = x^2 + 5$

Complete: $\frac{du}{dx} =$ $\therefore$ $du =$ dx

$\int x(x^2+5)^5 \, dx = \int$ du

$=$

$=$

Alternatively, could be done by inspection. Thinking of an answer and differentiating it to see how close you get - then adjusting your 'guess'.

Here using the method in the example. Again you need to replace xdx

Integrate.

Indefinite integral - so put the xs back in and don't forget the c.

110. $\int x^2 e^{x^3} \, dx$

Let $u = x^3$

Complete: $\frac{du}{dx} =$ $\therefore$ $du =$ dx

$\int x^2 e^{x^3} \, dx = \int$ du

$=$

$=$

Could be done by inspection.

Here using the method in the example. You need to replace x^2dx

You can check the answer by differentiating the result.

111. $\int \cos x \, e^{\sin x} \, dx$

Use $u = \sin x$.

112. $\int \cos x \sin^3 x \, dx$	*Rewrite* $\sin^3 x$ *as* $(\sin x)^3$. *Then let* $u = \sin x$.
113. $\int_0^{\pi/2} \sin x \cos^4 x \, dx$	*Rewrite* $\cos^4 x$ *as* $(\cos x)^4$. *Then let* $u = \cos x$ *Definite integral – so change the limits.* $x = 0 \quad u = \cos 0 = 1$ $x =$ *etc* *(You could obtain the integral in terms of x and substitute the x limits.)*

f′(x) / f(x):

Integrals where the top is the differential of the bottom. Use the result

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \ln\left| f(x) \right| + c}$$

Alternatively you can use the method of substitution.
Remember: ln is $\log_e$

114. Find $\int \frac{2x}{x^2+4} dx$	*Here the top is exactly the differential of the bottom. Just use the rule.*
115. Find $\int \frac{\sin x}{1-\cos x} dx$	*Use* $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$ *OR use substitution. Let* $u = 1 - \cos x$
116. Complete: $\int \frac{x^2}{x^3+4} dx = \frac{1}{3}\int \frac{3x^2}{x^3+4} dx$ $=$	*First make the top the differential of the bottom by multiplying the top by 3 (You can only do this with constants.) Then make it equal by dividing by 3 - but as it is a constant put it outside the integral. Use the rule.*
117. Complete: $\int \frac{3x}{x^2+1} dx = 3\int \frac{x}{x^2+1} dx$ $=$	*First take out the constant.* *Now make the top the differential of the bottom by multiplying the top by 2. Then make it equal by dividing by 2 - but as it is a constant put it outside the integral. Use the rule.*
118. Find $\int \frac{x+1}{x^2+2x} dx$	119. Find $\int \frac{2x^2}{x^3+1} dx$

Sometimes solutions which contain ln can be made simpler. e.g.

$$\int \frac{2x}{1+x^2} dx = \ln\left|1 + x^2\right| + c$$

$= \ln\left|1 + x^2\right| + \ln A$ Changing the constant to the **ln** of a different constant. e.g. 2.14 can be written $\ln 8.5$

$= \ln A\left|1 + x^2\right|$ Using a log law.

Integration by parts:

$$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx}$$

Example: $\int x e^x dx$ let $u = x$ and $\frac{dv}{dx} = e^x$ so $\frac{du}{dx} = 1$ and $v = e^x$ $\int x e^x dx = x e^x - \int e^x .1 dx$ $= x e^x - e^x + c$	Let $u = x$ and $\frac{dv}{dx} = e^x$ *Differentiate x and integrate* e^x *Then apply the rule.*
Example: $\int x \sin x \; dx$ let $u = x$ and $\frac{dv}{dx} = \sin x$ so $\frac{du}{dx} = 1$ and $v = -\cos x$ $\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \int \cos x \, dx$ $= -x \cos x + \sin x + c$	Let $u = x$ and $\frac{dv}{dx} = \sin x$ *Differentiate x and integrate* $\sin x$ *Then apply the rule.*
120. $\int x e^{-x} \; dx$	Let $u = x$ and $\frac{dv}{dx} = e^{-x}$ *Differentiate x and integrate* e^{-x} *Then apply the rule.*
121. $\int (x+1) e^x \; dx$	Let $u = x+1$ and $\frac{dv}{dx} = e^x$ *Differentiate x+1 and* $\sin x$ *integrate* e^{-x} *Then apply the rule.* *Simplify your answer.*
122. $\int 2x \cos 3x \; dx$	*You do this one.*
123. $\int \ln x \; dx$	*Special case. Rewrite* $\ln x$ *as* $1 \times \ln x$ *and apply integration by parts. You may apply the u and dv the wrong way round. (you will soon know as you will need to integrate* $\ln x$ *!)* *LEARN HOW TO DO THIS ONE.*
124. $\int x^2 \sin x \, dx$	*Use integration by parts twice. After the first application you will end up with* $\int ax \cos x$ *in your answer - then apply again.* *(Be careful with your signs.)*

125. $\int x^2e^x\ dx$	*Use integration by parts twice. After the first application you will end up with* $\int axe^x dx$ *in your answer - then apply again. (Be careful with your signs.)*
126. $\int_0^{\pi/2} 2x\sin 2x\ dx$	*Remember you are working in radians. (Otherwise you cannot use the results for the integration of* $\sin x$ *or* $\cos x$*).* *Work out as before then substitute in the limits to the whole answer.* *(No need for + c)*
127. $\int_0^1 10x e^{5x}\ dx$	*Work out as before then substitute in the limits to the whole answer.* *(No need for + c)*
128. $\int e^x \sin x\ dx$	*Use integration by parts twice. After the second application you will end up with* $-\int e^x \sin x\ dx$ *on the right hand side. Just add to both sides and divide throughout by 2.*

Integration using partial fractions:

You know from your work on integration

$$\int \frac{a}{x+c}\,dx = a\ln|x+c| + d \quad\text{and}\quad \int \frac{a}{b(x+c)}\,dx = \frac{a}{b}\int \frac{1}{x+c}\,dx = \frac{a}{b}\ln|x+c| + d \quad\text{and}\quad \int \frac{a}{bx+c}\,dx = \frac{a\ln|bx+c|}{b} + d$$

Denominator **must** be linear.

So
$$\begin{aligned}\int \frac{1}{x+2}\,dx &= \ln|x+2| + c\\ &= \ln|x+2| + \ln A\\ &= \ln A|x+2|\end{aligned}$$

and
$$\begin{aligned}\int \frac{5}{3(2x-1)}\,dx &= \frac{5}{3}\int \frac{1}{(2x-1)}\,dx\\ &= \frac{5}{3\times 2}\ln|2x-1| + c\\ &= \frac{5}{6}\ln|2x-1| + \ln A\\ &= \ln(2x-1)^{5/6} + \ln A\\ &= \ln A(2x-1)^{\frac{5}{6}}\end{aligned}$$

135. $\int \frac{2}{1-x}\,dx =$	136. $\int \sqrt{2x+1}\,dx =$
137. $\int \frac{2}{(1-x)^4}\,dx =$	138. $\int \frac{5}{(1-2x)^2}\,dx =$

139. Use the substitution u $= x^2 - 2$ to evaluate $\int x(x^2-2)^4\,dx$

140. Use the substitution u $= 2 + e^x$ to evaluate $\int e^x\sqrt{2+e^x}\,dx =$

141. Evaluate $\int_1^2 (2x-1)^3\,dx$

142. Use the substitution u $= \ln x$ to evaluate $\int_1^2 \frac{1}{x}\ln x\,dx$

143. Evaluate $\int x\cos x\,dx$

144. Express as $\frac{5x^2+3}{(x-1)(x+1)^2}$ the sum of three partial fractions. Hence evaluate $\int \frac{5x^2+3}{(x-1)(x+1)^2}dx$

129. Express $\frac{2x}{(x+2)(x-3)}$ in partial fractions and hence find $\int \frac{2x}{(x+2)(x-3)}\,dx$

130. Express $\frac{2x-5}{(x+2)(x-1)^2}$ in partial fractions and hence find $\int \frac{2x-5}{(x+2)(x-1)^2}\,dx$

(Note: one term is not linear in the denominator and so does not give ln)

Mixed questions 3

There are many different integrals covered in this course.

- Standard integrals.
- Definite integrals.
- Integration by using trigonometric identities.
- Integration using partial fractions.
- Integration by substitution.
- Integration of fractions where the top is the differential of the bottom.
- Integration by parts.
- Integration as a limit of a sum.
- Numerical integration - trapezium rule.
- Integrating parametric equations - page 37.

Try this mix of questions:

131. $\int 6x^2 + 2x - 1\ dx =$	132. $\int 5e^{2x}\ dx =$
133. $\int \frac{3}{2x}\,dx =$	134. $\int \frac{2}{3x-2}\ dx =$

145. Evaluate $\int_1^2 \frac{x^2}{x^3+3}\,dx$

146. Using the trapezium rule with 3 ordinates find an estimate for $\int_1^2 \log_{10} x\ dx$ to 3 significant figures.

Is your answer an under-estimate or over-estimate of the actual value? (Give reason.)

147. $y = (3x-5)^4$

Calculate $\lim\limits_{\delta x \to 0} \sum\limits_{x=1}^{x=2} y\,\delta x$

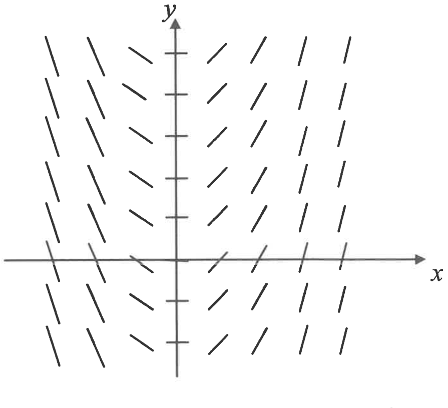
148. Sketch the curves $y = x^2 - 5x + 6$ and $y = 3x - x^2$

Find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 3x - x^2$

DIFFERENTIAL EQUATIONS

Differential equations contain $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ etc.

First order differential equations:
These are differential equations that contain only first differentials
e.g. $\frac{dy}{dx} = xy$, $\frac{dy}{dx} = x^2$, etc.
They tell you the gradient of the curve at different points on a grid. You can plot the gradients as shown. The grid gives you an idea of the shape of the family of solutions of the differential equation.



Differential equations - variables separable:

By multiplying and dividing get the y parts and dy on one side and the x parts and dx on the other (dy and dx must be on the top if you end up with fractions). Learn the method.

Example: Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x}{y^2}$$

$$y^2\frac{dy}{dx} = x$$

By multiplying or dividing - rearrange so that the ys are with the $\frac{dy}{dx}$ and the xs are on the other side. Integrate with respect to x.

$$\int y^2\frac{dy}{dx}dx = \int x\ dx$$

$\int y^2\frac{dy}{dx}dx$ is the same as $\int y^2 dy$

$$\int y^2 dy = \int x\ dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} + c$$

This is the *general solution.*

Find the particular solution if $y = 3$ when $x = 4$

$$\frac{27}{3} = \frac{16}{2} + c \quad \Rightarrow \quad c = 1$$

$$\frac{y^3}{3} = \frac{x^2}{2} + 1 \quad \Rightarrow \quad 2y^3 = 3x^2 + 6$$

This is a *particular solution.*

Note: The differential of y^2 with respect to x is

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

So the integral of $2y\frac{dy}{dx}$ with respect to x is y^2 or $\int 2y\frac{dy}{dx}dx$ is the same as $\int 2y\,dy = y^2$

Similarly for all functions of y. So here

$\int y^2\frac{dy}{dx}dx$ is the same as $\int y^2 dy$

Don't forget the + c

Substitute in the values of x and y to find c.

Write out the equation with this value of c.

---

149. Show that the general solution of $\frac{dy}{dx} = 2xy$ is $y = Ae^{x^2}$ where A is a constant.

Separate the variables and put in the integral sign.

Integrate. Remember the + c

Use the definition of logs to rewrite.

Use indices laws to rewrite.

e^c is a constant so rewrite as A.

---

150. $\frac{dy}{dx} = 2x^2 + 5$ Find the particular solution, given that $y = 10$ when $x = 3$

Separate the variables and put in the integral sign. Then just integrate.

Remember the + c

Substitute in the values of x and y to find c.

Write out the equation with this value of c.

---

151. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$

Separate the variables and put in the integral sign.

Integrate. Remember the + c

Rewrite the + c as + ln A

Use the log laws to rewrite the right hand side.

'Undo' the log to obtain a simple equation.

Forming and using differential equations may involve chain rule (as met before).
Here is one involving rates of change.

152. The area of a square is increasing at a rate that is proportional to the length of its side. Find the rate of change of its side. Complete: Area, A =	*A in terms of the side length s.*
$\frac{dA}{ds} =$	*Differentiate.*
$\frac{dA}{dt} \propto$	*The proportional part of the question.*
$\frac{dA}{dt} =$	*Write down = and put in a constant of proportionality.*
$\frac{ds}{dt} =$	*Chain rule.*

Note:

You use 'separating the variables' when you have a rate of change and you wish to find an equation and
you use the 'chain rule' when you are given a rate of change and you have to find another rate of change.

Exponential growth and decay:

Exponential growth and decay questions may require you to solve a differential equation.
Remember if a question says the rate of change of m (say) then you should write down $\frac{dm}{dt}$.

eg Solve the differential equation $\frac{dP}{dt} = kP$

	$\int \frac{1}{P}\, dP = \int k \; dt$	
	$\ln P = kt + \text{c}$	Don't forget the constant of integration.
	$P = e^{kt+\text{c}}$	Using the definition for log.
	$P = e^{kt} e^{\text{c}}$	Using indices law.
This is exponential growth or decay	$P = \text{A}\, e^{kt}$	Writing the constant e^{c} as a different constant A.

The constant term above, c, could have been written as a $\log_e$ i.e. $\ln \text{A}$ and then combined with the $\ln P$ to give $\ln \frac{P}{A} = kt$. You can then use the definition for logs to obtain the above answer.
You could be required to find c and k in ln form - you would then substitute given values in the second line.

153. Find x in terms of t given that $\frac{dx}{dt} = -kx$	*As the example above but with a minus to confuse you! This is exponential decay.* *You would use a minus if you were given a question where it told you that there was decay/decomposition/ decomposition/loss etc.*
154. Find r in terms of t given $\frac{dr}{dt} = kr$	*As the example above but with different variables.*

155. Solve the differential equation $\frac{d\theta}{dt} = -k\,\theta$ given that initially $\theta = 100$ and $\theta = 50$ when $t = 30$. State the value of k to 3 decimal places.

Initially is when t = 0

Here you are given some information to enable you to find the constant of integration and k.
Initially means when t = 0

156. Assume that the rate of increase in the population of a city after t years is proportional to the population at that time. If the population was 50 000 in 1950 and 100 000 in 2000 what will it be in 2020? In what year will it be 200 000?

157. Assume that the difference, x, between the temperature of a body and that of the surrounding air decreases at a rate that is proportional to this difference. If $x = 100^\circ$ when $t = 0$ minutes and $x = 60^\circ$ when $t = 20$ minutes, find x in terms of t.
Also find (i) x when $t = 40$ minutes (ii) t when $x = 30^\circ$

PARAMETRIC EQUATIONS

Sometimes it is convenient, or easier, to express x and y in terms of a third variable, called a parameter.
e.g. $x = t^2$, $y = 2t - 1$ These are called *parametric equations*.

Eliminating the parameter:

You can eliminate the parameter to find the direct relationship between x and y.
e.g. $x = t^2$, $y = 2t + 1$

so $t = \pm\sqrt{x}$ OR $t = \frac{y-1}{2}$

$\therefore y = \pm 2\sqrt{x} + 1$ $\therefore x = \left(\frac{y-1}{2}\right)^2 \Rightarrow 4x = (y-1)^2$

If the parametric equations use trig expressions, you may need to use trig identities to eliminate the parameter.
e.g. $x = 2\sin\theta$ $y = 3\cos\theta$ Where θ is the parameter.

$\therefore$ $\sin\theta = \frac{x}{2}$ $\cos\theta = \frac{y}{3}$

Using $\sin^2\theta + \cos^2\theta = 1$ you obtain

$$\left(\tfrac{x}{2}\right)^2 + \left(\tfrac{y}{3}\right)^2 = 1$$

or $\frac{x^2}{4} + \frac{y^2}{9} = 1$ Multiply through by 4 and 9.

$9x^2 + 4y^2 = 36$ Which is the Cartesian equation of the curve.

Find the Cartesian equation of the curve whose parametric equations are:

158. $x = 2t^2$ $y = 5t$	159. $x = \frac{4}{t}$ $y = t$
160. $x = 3\cos\theta$ $y = 3\sin\theta$	161. $x = e^t$ $y = \cos t$

Sketching the curve:

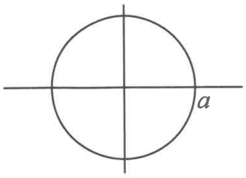
Plot a number of points for various values of t.
e.g. $x = t^2$, $y = 2t - 1$
Complete:

t	-2	-1	0	1	2
x	4	1			
y	-5	-3			

Plot the points (and label t = 1, t = 2 etc) and hence sketch the curve.

Parametric equations of circles centred on the origin.

For a circle whose centre is the origin and radius is a
the Cartesian of this circle is $x^2 + y^2 = a^2$
the parametric equations of this circle are $x = a\cos\theta$ $y = a\sin\theta$



Differentiating parametric equations:

Use $\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad OR \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}}$

e.g. $x = t^2$, $y = 3t - 1$

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3$

So $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ *OR* $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$

$= 3 \times \frac{1}{2t}$ $= 3/2t$

$= \frac{3}{2t}$

162. Find $\frac{dy}{dx}$ in terms of t, if

Complete: (a) $x = 3t$, $y = 2t^2$

$\frac{dx}{dt} =$, $\frac{dy}{dt} =$

$\frac{dy}{dx} =$

(b) $x = t^2$, $y = 2t^3$

$\frac{dx}{dt} =$, $\frac{dy}{dt} =$

$\frac{dy}{dx} =$

(c) $x = 2\cos t$, $y = 3\sin t$

$\frac{dx}{dt} =$, $\frac{dy}{dt} =$

$\frac{dy}{dx} =$

(d) $x = \frac{1}{t}$, $y = 2t^2$

$\frac{dx}{dt} =$, $\frac{dy}{dt} =$

$\frac{dy}{dx} =$

Differentiate both with respect to t.

Use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ *OR* $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$
and tidy up.

163. Find the equation of the tangent to the curve given by $x = t^2$, $y = 5t$ at the point where $t = 1$

Differentiate both with respect to t.
Use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ *OR* $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$
and tidy up.

Find the gradient of the tangent at $t = 1$.

Use $y - y_1 = m(x - x_1)$ *or* $y = mx + c$ *to find the equation of the tangent.*

164. Find the turning points of the curve given by $x = t$, $y = t^3 - 3t$, and distinguish between them.

Differentiate both with respect to t.
Use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ *OR* $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$
and tidy up.

Find the values of x that make $\frac{dy}{dx} = 0$.

To find max, or min, look at $\frac{dy}{dx}$ *just before, and just after, the turning point OR differentiate a second time to find* $\frac{d^2y}{dx^2} = \left(\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}\right)$ *and using the values of t obtained earlier to determine max or min.*

165. Find the Cartesian equation of the curve given by $x = 2\cos\theta$, $y = 5\sin\theta$

166. Find the equation of the normal to the curve given by $x = 5\cos\theta$, $y = 5\sin\theta$ at the point where $\theta = \frac{\pi}{3}$ Write down the Cartesian equation of the curve.

Integrating parametric equations:

If a curve is given parametrically and you are required to find the area between the curve and the x axis then the following can be used:

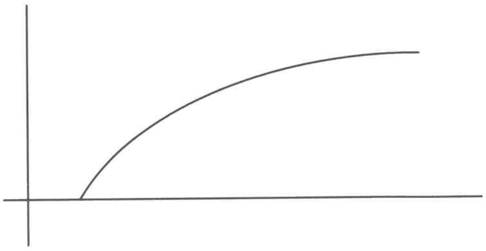
$$A = \int y\ dx$$

$$A = \int y \frac{dx}{dt}\ dt$$

e.g. Find the area between the curve given by $x = t^2$, $y = 2t - 1$ for $\frac{1}{2} \le t \le 2$ and the x axis.

First sketch the curve

t	0.5	1	1.5	2
x	0.25	1	2.25	4
y	0	1	2	3



$$\frac{dx}{dt} = 2t$$

$$A = \int_{0.5}^{2} y \tfrac{dx}{dt}\, dt$$

$$= \int_{0.5}^{2} (2t-1)2t\ dt$$

$$= \int_{0.5}^{2} 4t^2 - 2t\ dt$$

$$= \left[\tfrac{4t^3}{3} - t^2\right]_{0.5}^{2}$$

$$= (\tfrac{32}{3} - 4) - (\tfrac{1}{6} - \tfrac{1}{4})$$

$$= \tfrac{20}{3} - \tfrac{1}{6} + \tfrac{1}{4} = 6\tfrac{3}{4} \textit{ square units}$$

167. Find the area between the curve given by $x = 4\cos\theta$, $y = 2\sin\theta$ for $0 \le x \le 4$ and the x axis.

Sketch the curve (you may recognise the type of curve).

Careful with the limits.
x between 0 and 4 correspond to θ between $\frac{\pi}{2}$ and 0 (in that order).

Use the form above.

Use the identity for $\cos 2\theta$ to change the $\sin^2\theta$

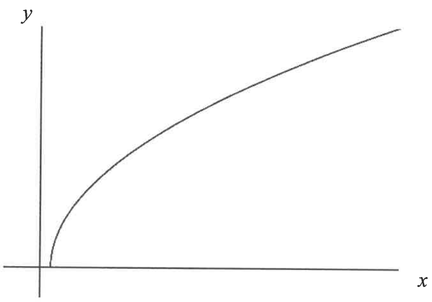
Integrate and substitute in the limits.

Parametric equations are often used in motion work.
The position of an object (x, y) at t seconds is given by
$x = t^2 + 1\ ,\ y = 4t\ \ ,\ t \geq 0$

Plotting points when t = 0, 1, 2, 3, 4, 5 gives the curve shown.

Eliminating the parameter gives

$y = 4\sqrt{x-1}$ (positive root since $t \geq 0$)



168. A student throws a 5 kg weight from a height of 1.8 m with an initial speed of 5 ms^{-1} at an angle of 45°. The equations for the path are $x = (5\cos 45^\circ)t$, $y = 1.8 + (40\sin 45^\circ)t - 5t^2$ for $t \geq 0$
Assuming the ground is level how far does the student throw the weight.

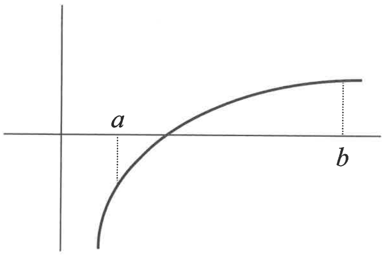
169. An object is moving with constant velocity at point P(8, 3) at t = 0 At t = 5 its position is (−7, 13)
(a) What is the horizontal and vertical velocity of the object?
(b) What are the parametric equations that describe the motion of the object?

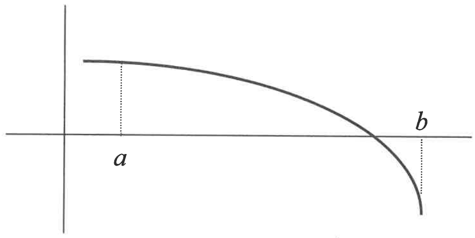
NUMERICAL METHODS

Finding roots:

Not all equations can be solved using algebraic methods e.g. factorising, or using a formula like the quadratic formula.

To show that roots (solutions) of equations lie between two values:





*When the interval is **too large**, the sign may not change as there may be an even number of roots. i.e. the curve cuts the axis twice (or four times etc) within the interval.*
*If the function is **not continuous**, the sign may change but there may be an asymptote (not a root).*

If a graph is continuous for $a \leq x \leq b$ and f(x) **changes sign** between $x = a$ and $x = b$ then there is a root (solution) of **f(*x*) = 0** between $x = a$ and $x = b$

170. Find two consecutive integers between which the root of
$x^3 - 12x^2 + 48x - 61 = 0$ lies.

$f(x) = x^3 - 12x^2 + 48x - 61$

Complete:

f(0) = f(1) =

f(2) = f(3) =

Change of sign ∴ root lies between $x =$ and $x =$

Substitute 0 for x and evaluate.
Substitute 1 for x and evaluate.

Substitute 2 for x and evaluate.
Substitute 3 for x and evaluate.

171. Show that the equation $x^4+x^3-4x^2-5=0$ has a root between $x=-2.7$ and $x=-2.8$.	
$f(x)=x^4+x^3-4x^2-5$	
$f(-2.7)=$	*Substitute –2.7 for x and evaluate.*
$f(-2.8)=$	*Substitute –2.8 for x and evaluate.*
Change of sign ∴ root lies between $x=-2.7$ and $x=-2.8$	

172. Show that the point of intersection of $y=2x+2$ and $y=x^3+3x^2-7$ lies between $x=1.6$ and $x=1.7$	*At point of intersection y coordinates are the same. So just put the right hand sides equal.* *Rearrange to give = 0* *Substitute 1.6 for x and evaluate.* *Substitute 1.7 for x and evaluate.* *State that there is a change of sign and therefore the x coordinate of the point of intersection lies between these values of x.*

Simple iterative methods:

If you know a point near to a solution, then you can use numerical methods to find an approximate solution.

$x = g(x)$ method:

This method can be used to find the root of an equation $f(x) = 0$ which can be written in the form $x = g(x)$

The equation $x^3-2x-3=0$ can be rearranged in different ways to obtain $x =$ ....

e.g. $x=\frac{1}{2}(x^3-3)$

$x=\sqrt[3]{2x+3}$

$x=\frac{2x+3}{x^2}$

$x=\frac{3}{x^2-2}$

Make sure that you can rearrange the given equation $x^3-2x-3=0$ into all the forms given here.

Some of these equations may be used to give an iterative formula, that will converge to a root when given a value near to the root. (Note: Some equations may give an iterative formula that will not converge to a root.)

173. (i) Rearrange $e^x+x-5=0$ into the form $x=\ln(f(x))$	*Rearrange to give $e^x =$* *Take logs base e of both sides.* *Use a log law and ln e = 1 to obtain x =*
(ii) Use the iteration $x_{n+1}=\ln(f(x_n))$ based on (i) with $x_1 = 1$ to find x_2, x_3 and x_4. Give your answers to 3 decimal places.	*Use $x_{n+1} = \ln(5-x_n)$* *Give answers to 3 d.p. but use the full value of x from your calculator in the next calculation.*
(iii) Check your result gives a solution of $e^x+x-5=0$ correct to 2 decimal places by looking for a change of sign and state your solution.	*State your solution.*

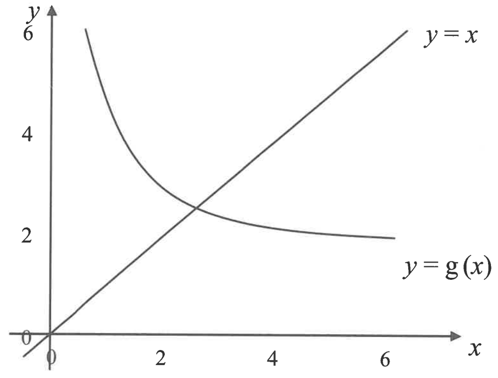
174. Use the iterative formula $x_{n+1} = \frac{4}{x_n^{2}} + \frac{2x_n}{3}$ with $x_1 = 2$ to obtain x_2, x_3 and x_4. If the sequence generated by this formula converges to a value α, find the equation in the form $ax^3 + bx^2 + cx + d = 0$ which is satisfied by α

The real roots of $x = g(x)$ (and hence $f(x) = 0$) are where the lines $y = x$ and $y = g(x)$ intersect. You can draw cobweb/staircase diagrams to show how the iterations converge on a root, or diverge from a root.

175. For the following draw cobweb/staircase diagrams and describe their behaviour.

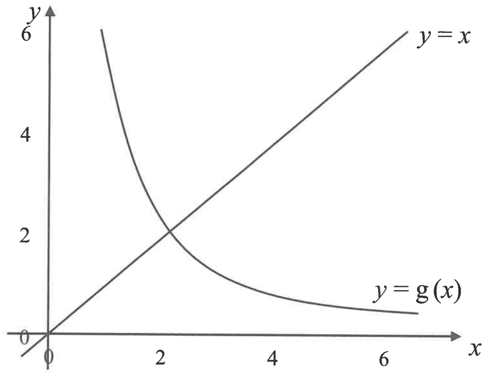
From the starting value (your first value of x) go vertically to the curve (equivalent to substituting the value for x into the right hand side of the equation) and then horizontally to y = x (this gives the second value for x). Then vertically to the curve (again equivalent to the substitution of this value) and then horizontally to y = x (for the third value of x) and keep repeating.

(a)



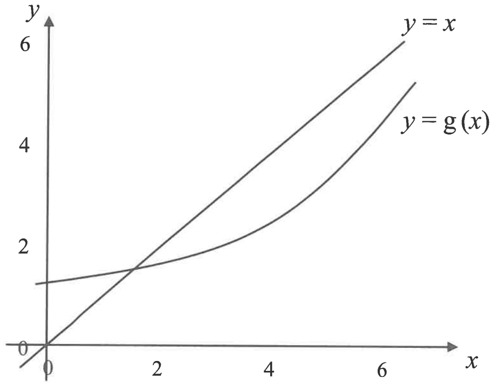
Starting at x = 1

(b)



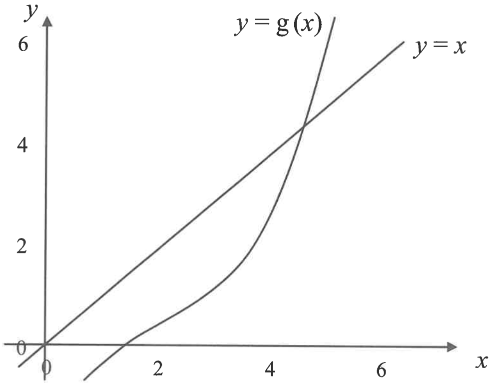
Starting at x =2

(c)



Starting at x = 5

(d)

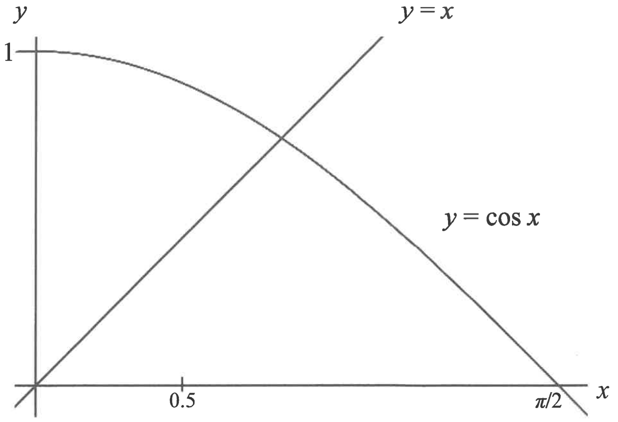


Starting at x = 4

(i) What property of $g(x)$ determines whether you have a staircase or cobweb diagram?

(ii) Explain how the gradient of $g(x)$ determines whether an iteration based on $x_{n+1} = g(x_n)$ will converge or not.

176. The sketch graphs show that the equation $x = \cos x$, where the angle is measured in radians, has one positive root. On this diagram show how an initial value of $x_1 = 0.5$ will converge on this root when the iterative formula $x_{n+1} = \cos x_n$ is used. Find this root to 1 decimal place.



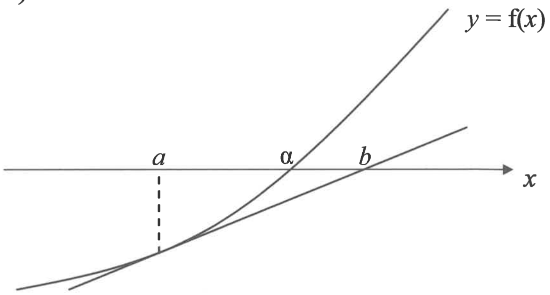
Newton - Raphson method:

Consider the equation $f(x) = 0$ that has a root α and that a is an approximation for α.
The curve $y = f(x)$ cuts the x axis at α. The tangent to the curve at $x = a$ cuts the x axis at b. In most circumstances b will be closer to α and therefore a better approximation to the root, α.

Equation ot the tangent at a is $y - f(a) = f'(a)\,(x - a)$
This cuts the x axis where $y = 0$
i.e. where $x = a - f(a) / f'(a)$

So if a is an approximation to a root of $f(x) = 0$

Then $\boxed{b = a - \dfrac{f(a)}{f'(a)}}$ is a better approximation.

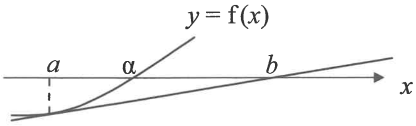


You may be used to using the iterative form:

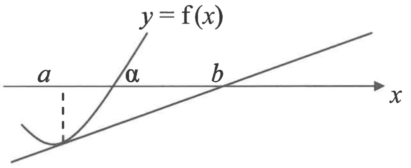
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Failure of the Newton-Raphson method will occur when:

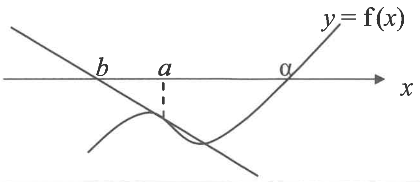
(a) $f'(a)$ is too small.



(b) $f'(a)$ increases too quickly.



(c) $f'(a)$ is too far from the root.



177. By sketching $y = e^x$ and $y = 3 - x$ show that $e^x = 3 - x$ has only one root. Use the Newton-Raphson method to find this root to three decimal places starting with the first approximation, $x_1 = 0.5$

Sketch $y = e^x$ and $y = 2 - x$

Rearrange $e^x = 2 - x$ to equal 0 and write down $f(x)$.
Work out $f'(x)$.

Use $x_{n+1} = x_n - f(x_n) / f'(x_n)$ to obtain a better approximation. Call this x_2
Repeat to required degree of accuracy.
Check the boundaries for your approximation have a root between them. i.e. change of sign.

178. Show that $f(x) = x^3 - 9x + 12$ has a solution between $x = -3$ and $x = -4$ Using a first approximation of -4 and the Newton-Raphson method find this root to three decimal places.

179. Show that $f(x) = \cos 2x - x$ has a solution between $x = 0.5$ and $x = 0.6$ Using a first approximation of 0.5 and the Newton-Raphson method find this root to three decimal places.

Numerical Integration - The Trapezium Rule:

You can find an estimate for an integral by considering trapeziums (only use this method if told to do so).

$$\int_a^b f(x)\,dx = \tfrac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots\ldots.. + 2y_{n-1} + y_n\right)$$ where $h = \frac{b-a}{n}$ for n trapeziums

Remember: 3 trapeziums gives 4 ordinates y_0 , y_1 , y_2 , y_3
4 trapeziums gives 5 ordinates y_0 , y_1 , y_2 , y_3 , y_4
etc.

180. Find the approximate value of $\int_1^2 \sqrt{(x^2+1)}\,dx$ using the trapezium rule with four trapeziums.

Complete: $x_0 = 1$ $y_0 = \sqrt{(1^2+1)} =$

$x_1 =$ $y_1 =$

$x_2 =$ $y_2 =$

$x_3 =$ $y_3 =$

$x_4 =$ $y_4 =$

h =

$\int_1^2 \sqrt{(x^2+1)}\,dx \approx$

1 *2*

Four trapeziums so five ordinates. Work out the five x coordinates between 1 and 2 - the line above may help. Work out the corresponding y coordinates.

Work out h.

Use the trapezium rule to give an estimate for the value of the integral.

For trig functions - work in radians.

181. Use the trapezium rule with 3 trapeziums to find an estimate for the area between the curve $y = \sin^3 x$ the x axis and the line $x = \frac{\pi}{2}$ Give your answer to 2 decimal places.

A sketch of $y = \sin^3 x$ will help.

Use $V = \int_a^b y\,dx$

Four ordinates.

You need to find $\int_0^{\pi/2} \sin^3 x\,dx$ *so estimate the integral using the trapezium rule with* $y = \sin^3 x$

Work out the four x coordinates between 0 and $\frac{\pi}{2}$. Work out the corresponding y coordinates.
Work out h.

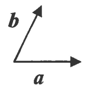
Use the trapezium rule to give an estimate for the value of the integral.

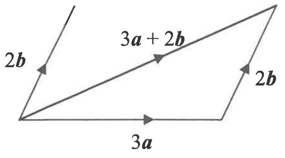
182. Using the trapezium rule with 3 ordinates find an estimate for $\int_1^2 \ln x\,dx$ to 3 significant figures. Is your answer an under-estimate or over-estimate of the actual value? (Give a reason.)

VECTORS

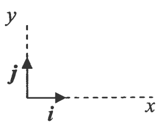
Vectors in three dimensions: All the work on vectors in book 2 can be extended to three dimensions.

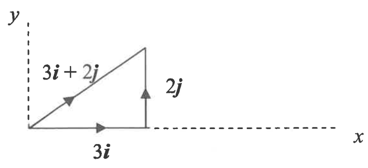
Two dimensions.





Taking unit vectors in the x and y directions.

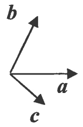


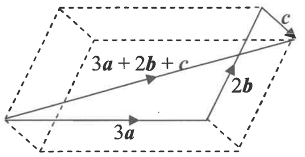


Modulus or magnitude of vector $3\boldsymbol{i} + 2\boldsymbol{j}$

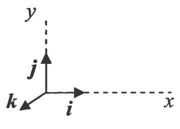
$$|3\boldsymbol{i} + 2\boldsymbol{j}| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

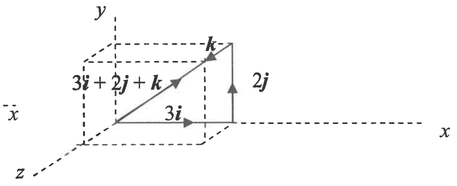
Three dimensions.





Taking unit vectors in the x, y and z directions.





Modulus or magnitude of vector $3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$$|3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

Two dimensions. Alternative notation: $3\boldsymbol{i}+2\boldsymbol{j}$ can be written $\begin{pmatrix}3\\2\end{pmatrix}$ *Examples: Velocity vector* $(3\boldsymbol{i}+4\boldsymbol{j})\ \text{ms}^{-1}$ Speed = $\sqrt{3^2+4^2} = \sqrt{25} = 5\ \text{ms}^{-1}$ *Force vector* $(5\boldsymbol{i}+12\boldsymbol{j})$ Newtons *Magnitude of force* = $\sqrt{5^2+12^2} = \sqrt{169} = 13$ N Resultant vectors: $\boldsymbol{a} = 3\boldsymbol{i}+2\boldsymbol{j} \quad \boldsymbol{b} = \boldsymbol{i}-4\boldsymbol{j}$ $2\boldsymbol{a}+4\boldsymbol{b} = 2\times(3\boldsymbol{i}+2\boldsymbol{j})+4\times(\boldsymbol{i}-4\boldsymbol{j})$ $= 10\boldsymbol{i}-12\boldsymbol{j}$ Position vectors: *These are vectors that start at the origin.* *(3, 5) has position vector* $3\boldsymbol{i}+5\boldsymbol{j}$	*Three dimensions.* Alternative notation: $3\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ can be written $\begin{pmatrix}3\\2\\1\end{pmatrix}$ *Examples: Velocity vector* $(3\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k})\ \text{ms}^{-1}$ Speed = $\sqrt{3^2+4^2+5^2} = \sqrt{50}\ \text{ms}^{-1}$ *Force vector* $(\boldsymbol{i}+3\boldsymbol{j}+7\boldsymbol{k})$ Newtons *Magnitude of force* = $\sqrt{1^2+4^2+5^2} = \sqrt{42}$ N Resultant vectors: $\boldsymbol{a} = 3\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k} \quad \boldsymbol{b} = \boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$ $2\boldsymbol{a}+4\boldsymbol{b} = 2\times(3\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})+4\times(\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k})$ $= 10\boldsymbol{i}-12\boldsymbol{j}+6\boldsymbol{k}$ Position vectors: *These are vectors that start at the origin.* *(3, 5, 2) has position vector* $3\boldsymbol{i}+5\boldsymbol{j}+2\boldsymbol{k}$

You write vector a as $\underline{a}$ or $\overrightarrow{OA}$ and the magnitude as a, $|\underline{a}|$, OA or $|\overrightarrow{OA}|$

Finding the distance between two points:
Find the vector between the two points and then work out the magnitude of this vector.

183. Points A and B are (1, 2, 3) and (2, 1, 1) Find the distance between the two points to three significant figures.

A sketch helps - just any two points A and B and an origin O.
Write down the position vector $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$.
Work out $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

(Remember: $\overrightarrow{AO} = -\overrightarrow{OA}$ *)*

Work out the magnitude of $\overrightarrow{AB}$ *to give the required distance.*

Finding the point dividing a line in a given ratio:

184. Points A and B are (1, 6, 3) and (6, – 4, 8) If P divides AB in the ratio 3:2 find the coordinates of P.

A sketch helps - just any two points A and B and an origin O.
Write down the position vector $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$.

Work out $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

Work out $\overrightarrow{AP}$ *using the ratio.*

Work out $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$
and hence the point P.

Finding the angle between two vectors:

185. Points A, B and C are (1, 1, 3), (5, 3, 4) and (2, 1, –3) Find the angle ACB.

A sketch helps - just any three points A, B, C and an origin O.
Write down the position vector $\overrightarrow{OA}$, $\overrightarrow{OB}$ *and* $\overrightarrow{OC}$.

Work out $\overrightarrow{AB}$, $\overrightarrow{AC}$ *and* $\overrightarrow{BC}$.

Work out the magnitudes of $\overrightarrow{AB}$, $\overrightarrow{AC}$ *and* $\overrightarrow{BC}$ *and use the cosine rule to work out the required angle.*

PROOF

Proof by contradiction:

One well-known use of this method is in the proof that $\sqrt{2}$ is irrational.

Rational numbers are those which can be written in fractions, that is as one integer divided by another. They can be put into what is called *irreducible form* , which is where the numerator and denominator have no common factors other than 1 (they are then called coprime). Irrational numbers are those which cannot be put into such a form, such as π and $\sqrt{2}$.

Prove that $\sqrt{2}$ is irrational by contradiction.

Assume that the $\sqrt{2}$ is rational. Then the $\sqrt{2}$ can be written as $\frac{p}{q}$ where p and q are integers, and $\frac{p}{q}$ is reduced as much as possible i.e. p and q don't share any common factors other than 1 (they are coprime).

$$\sqrt{2} = \frac{p}{q}$$

Square both sides $\quad 2 = \frac{p^2}{q^2}$

$2q^2 = p^2 \quad \therefore p^2$ is a multiple of 2 and so it is even.

If p^2 is even then p is also even (As even×even = even and odd×odd = odd) and so p can be written as $2n$

So $\quad 2q^2 = (2n)^2 = 4n^2$

Dividing by 2 gives $\quad q^2 = 2n^2 \quad$ Which tells you that q^2 is even and therefore q is even.

If p and q are both even then they have 2 as a common factor, which contradicts the assumption that they do not have any common factors. Thus our assumption is incorrect, and $\sqrt{2}$ is not rational.

Prove that there are infinitely many prime numbers.

Suppose that there are a finite number of prime numbers where p_n is the largest prime number.
Then it would be possible to list all of the prime numbers in order $\quad p_1, p_2, p_3, \ldots.., p_n$

Consider the product of all of these prime numbers $\quad p_1 \times p_2 \times p_3 \times \ldots.. \times p_{n-1} \times p_n$
This number is divisible by all of the prime numbers.

Now, add 1 to this number: $p_1 \times p_2 \times p_3 \times \ldots.. \times p_{n-1} \times p_n + 1$
This number is **not** divisible by any of the prime numbers, because dividing it by any prime will always results in a remainder of 1. So this number is a prime number.
However, this number is larger than p_n
This contradicts the supposition that p_n is the largest prime number.
Therefore, there is no largest prime number. There are infinitely many prime numbers.

186. Prove, by contradiction, that if p^2 is even then p is even.	*You do this by considering a number p whose square, p^2, is even, and assuming that this p is **not** even. Then you try to arrive at a contradiction.*

187. Prove that the sum of a rational number and an irrational number, is irrational.

188. Prove that for every real number $x \in [0, \pi/2]$, $\sin x + \cos x \geq 1$

Mixed questions 4

189. (i) Show that $(3x + 2)$ is a factor of $9x^3 + 9x^2 - 4x - 4$ (ii) Simplify $\frac{9x^3 + 9x^2 - 4x - 4}{3x^2 + 5x + 2}$

190. Evaluate $\int_0^2 \frac{2x^2}{x^3 + 1} dx$

191. (i) Show that $\frac{d}{dx}(\tan x) = \sec^2 x$ (ii) Find $\int x \sec^2 x \, dx$

192. A spherical balloon is being inflated at a rate of 3 cm^3 per second. What is the rate of growth of the radius when r = 2 cm?

193. $f(x) = 2x^3 - x^2 - 13x - 6$ (a) Find $f(3)$
(b) Use the factor theorem to show that $(2x + 1)$ is a factor of $f(x)$ (c) Solve the equation $f(x) = 0$

194. $f(x) = \frac{8-5x}{(1-x)^2(4-x)}$ Express $f(x)$ in partial fractions and hence find $f'(x)$

195. Express $\frac{10x^3+13x^2-3x+4}{5x-1}$ in the form $ax^2 + bx + \frac{c}{5x-1}$, where a, b and c are integers.

196. Differentiate 10^x with respect to x

197. Find $\int \frac{2}{2x+1} + \frac{2}{(2x+1)^2} dx$

198. Find the point of inflection for the curve $y = \frac{x^3}{12} - \frac{x^2}{4} + \frac{x}{2}$ and show that it is a non-stationary point of inflection.

199. The position vectors of three points A, B and C are $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $-3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ respectively. Find the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Hence find the angle ABC.

200. Use the substitution $x = \tan\theta$ to find $\int \frac{1}{1+x^2}\, dx$

201. Find the gradient of the curve $3x^2 + 3y^2 - 5xy = 5$ at the point (1, 2)

202. If $x = 2\cos t \quad y = 4\sin^2 t$ find $\frac{dy}{dx}$ in terms of t

203. (a) Work out $\int \cos^2 x\, dx$ (b) Evaluate $\int_0^{\pi} 10x \sin x\, dx$

204. The rate, in $\text{cm}^3\,\text{s}^{-1}$, at which a liquid is released from a container at any time t seconds is proportional to the volume of the liquid $V\,\text{cm}^3$ in the container at any instant. At time $t = 0$, $V = A$
By forming and integrating a differential equation show that $V = A\,e^{-\lambda t}$, where λ is a positive constant. Given $V = \frac{3}{4}A$ at $t = 8$, find the value of t, to the nearest second, when $V = \frac{1}{2}A$

205. Prove, by contradiction, that for all values of $x > 0$, $x + \frac{1}{x} \geq 2$

206. Express $\frac{3-x}{2+x}$ in the form $A+\frac{B}{2+x}$

207. (a) Evaluate $\int_1^4 \sqrt{x}\, dx$ (b) Use the trapezium rule with three ordinates to estimate $\int_1^4 \sqrt{x}\, dx$

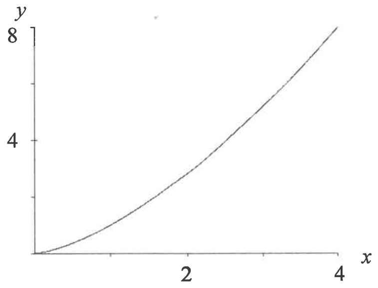
(c) Calculate the percentage error in using the trapezium rule with three ordinates to evaluate $\int_1^4 \sqrt{x}\, dx$

208. A Sketch of the curve with parametric equations $x = t^2$, $y = t^3$ for $t \geq 0$ is shown below.

(a) Find the equation of the tangent where $t = 2$.

(b) Find the region bound by the curve, the x axis and the tangent.

(c) Find where the line $3y = 2x + 1$ meets the curve.



209. Show that $x^3 + x - 3 = 0$ has a root in the interval $[1, 2]$. The iterative formula $x_{n+1} = \sqrt[3]{3 - x_n}$ converges on this root. Use the iterative formula with $x_1 = 1$ to find this root to 3 significant figures.

210. Express $\dfrac{4-x}{(2+x)(1-x)}$ in partial fractions. Hence write $\dfrac{4-x}{(2+x)(1-x)}$ as a series in ascending powers of x up to the term in x^3, stating the range of values of x for which the expansion is valid.

211. Find the particular solution in the form $y = \mathrm{f}(x)$ of the differential equation $\dfrac{dy}{dx} = y\sin 2x$ given that $y = 5$ when $x = \dfrac{\pi}{4}$

212. For A(1, 3, −1) and B(2, 4, 2). Find (i) $\overrightarrow{\mathrm{AB}}$ (ii) the angle OAB, where O is the origin.

213. By drawing two graphs show that the equation $e^{x} - \frac{1}{x} = 0$ has only one root and verify that this lies between 0.5 and 0.6 Find this root to three decimal places using the Newton-Raphson method with an initial approximation of 0.6

Answers

Page3

1. $\frac{1}{x}+\frac{1}{x+1} \equiv \frac{x+1}{x(x+1)} + \frac{x}{x(x+1)} \equiv \frac{x+1+x}{x(x+1)} = \frac{2x+1}{x(x+1)}$

2. $\frac{1}{x(x+1)} - \frac{3}{(x+1)(x+2)} \equiv \frac{x+2}{x(x+1)(x+2)} - \frac{3x}{x(x+1)(x+2)}$

$\equiv \frac{x+2-3x}{x(x+1)(x+2)} \equiv \frac{-2x+2}{x(x+1)(x+2)}$

3.

$\frac{2x}{(x+2)^2}+\frac{3}{x+2}+\frac{1}{x} \equiv \frac{2xx}{x(x+2)^2}+\frac{3x(x+2)}{x(x+2)^2}+\frac{(x+2)^2}{x(x+2)^2}$

$\equiv \frac{2x^2+3x^2+6x+x^2+4x+4}{x(x+2)^2} \equiv \frac{6x^2+10x+4}{x(x+2)^2}$

4. $2 + \frac{5}{x-4} \equiv \frac{2(x-4)}{x-4}+\frac{5}{x-4} \equiv \frac{2x-8+5}{(x-4)} \equiv \frac{2x-3}{(x-4)}$

Page4

5. $\text{f}(x)=x^3-11x-6 \quad \text{f}(-3)=-27+33-6=0$

$\therefore$ $(x+3)$ is a factor of $x^3-11x-6$

6. $(-1)^3-a(-1)+6=0 \Rightarrow -1+a+6=0$

$a=-5$

7. $2(-2)^3+a(-2)^2+b(-2)-2= 0$

$4a-2\text{b}=18$

$2(-1/2)^3+a(-1/2)^2+b(-1/2)-2= 0$

$a-2\text{b}=9$

$3a=9 \Rightarrow a=3\,, \quad b=-3$

8. $\text{f}(x)=4x^3+4x^2-x-1$

$\text{f}(-1)=-4+4+1-1=0 \quad \therefore (x+1)$ is a factor

$4x^3+4x^2-x-1=(x+1)(4x^2-1)$

$=(x+1)(2x+1)(2x-1)$

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9. $4x^2+3=\text{A}x(2x+1)+\text{B}(2x+1)+\text{C}$

$x=-½ \quad 4=\text{C}$

$x=0 \quad 3=\text{B}+4 \quad \Rightarrow \text{B}=-1$

$x=1 \quad 7=3\text{A}-3+4 \Rightarrow 6=3\text{A} \quad \Rightarrow \text{A}=2$

$\frac{4x^3+3}{2x+1} \equiv 2x-1+\frac{4}{2x+1}$

10. $2x-3=\text{A}(x+2)+\text{B}$

$x=-2 \quad -7=\text{B}$

$x=0 \quad -3=2\text{A}-7 \quad 4=2\text{A} \quad \Rightarrow \text{A}=2$

$\frac{2x-3}{x+2} \equiv 2-\frac{7}{x+2}$

Page6

11. $2x^2+2=\text{A}(x+1)(x-3)+\text{B}(x-3)+\text{C}(x+1)$

$x=-1, \quad 4=-4\text{B} \Rightarrow \text{B}=-1$

$x=3, \quad 20=4\text{C} \Rightarrow \text{C}=5$

$x=0,\ 2=-3\text{A}+3+5 \Rightarrow -6=-3\text{A} \Rightarrow \text{A}=2$

$\frac{2x^2+2}{(x+1)(x-3)} \equiv 2-\frac{1}{x+1}+\frac{5}{x-3}$

12. (a) $\text{f}(1)=3+5-4-4=0$

(b) $\text{f}(-2/3)=3(-8/27)+5(4/9)-4(-2/3)-4=0$

$\therefore (3x+2)$ is a factor

(c) $\text{f}(x)=(x-1)(3x+2)(x+2)$

Fraction $= \frac{(x-1)(3x+2)(x+2)}{(3x+2)(x-1)} = x+2$

13. (a) $\frac{1-t}{t-t^2}=\frac{1-t}{t(1-t)}=\frac{1}{t}$

(b) $\frac{2x^2+4x}{x+2}=\frac{2x(x+2)}{x+2}=2x$

14. $\frac{(2x+3)(2x(2x-1)+(2x-9))}{(2x+3)(2x+3)}=\frac{(2x+3)(4x^2-9)}{(2x+3)(2x+3)}$

$=\frac{(2x+3)(2x+3)(2x-3)}{(2x+3)(2x+3)}=2x-3$

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15. TypeI

$\equiv \frac{A(x-2)}{(x+1)(x-2)}+\frac{B(x+1)}{(x+1)(x-2)} \equiv \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$

So $2x-3 \equiv \text{A}(x-2)+\text{B}(x+1)$ This is true for all x

$x=2 \quad 1= 3\text{B} \quad \Rightarrow \quad \text{B}=1/3$

$x=-1 \quad -5=3\text{A} \quad \Rightarrow \quad \text{A}=5/3$

$\frac{2x-3}{(x+1)(x-2)} \equiv \frac{5}{3(x+1)}+\frac{1}{3(x-2)}$

16.

$\frac{2x+1}{(x-2)(x+4)} \equiv \frac{A}{x-2}+\frac{B}{x+4} \equiv \frac{A(x+4)+B(x-2)}{(x-2)(x+4)}$

So $2x+1 \equiv \text{A}(x+4)+\text{B}(x-2)$

$x=2 \quad 5=6\text{A} \quad \Rightarrow \quad \text{A}=5/6$

$x=-4 \quad -7=-6\text{B} \quad \Rightarrow \quad \text{B}=7/6$

$\frac{2x+1}{(x-2)(x+4)} \equiv \frac{5}{6(x-2)}+\frac{7}{6(x+4)}$

17. $\frac{4-5x}{(2x-1)(x-1)} \equiv \frac{A}{2x-1}+\frac{B}{x-1}$

$4-5x \equiv \text{A}(x-1)+\text{B}(2x-1)$

$x=1 \quad -1=\text{B}$

$x=½ \quad 3/2=-½\,\text{A} \Rightarrow \quad \text{A}=-3$

$\frac{4-5x}{(2x-1)(x-1)} \equiv -\frac{3}{2x-1}-\frac{1}{x-1}$

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18. Type II

$\equiv \frac{A(x-1)^2}{(x+3)(x-1)^2}+\frac{B(x+3)(x-1)}{(x+3)(x-1)^2}+\frac{C(x+3)}{(x+3)(x-1)^2}$

$\equiv \frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$

So $2x \equiv \text{A}(x-1)^2+\text{B}(x+3)(x-1)+\text{C}(x+3)$

This is true for all x

$x=1 \quad 2=4\text{C} \quad \Rightarrow \quad \text{C}=½$

$x=-3 \quad -6=16\text{A} \quad \Rightarrow \quad \text{A}=-6/16=-3/8$

$x=0 \quad 0=-3/8+\text{B}(-3)+3/2 \quad \Rightarrow \quad \text{B}=3/8$

$\frac{2x}{(x+3)(x-1)^2} \equiv \frac{-3}{8(x+3)}+\frac{3}{8(x-1)}+\frac{1}{2(x-1)^2}$

19. $\frac{x+1}{(x-1)(x+2)^2} \equiv \frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$

$\equiv \frac{A(x+2)^2+B(x-1)(x+2)+C(x-1)}{(x-1)(x+2)^2}$

$x+1=A(x+2)^2+B(x-1)(x+2)+C(x-1)$

$x=1 \quad 2=9A \quad \Rightarrow \quad A=\frac{2}{9}$

$x=-2 \quad -1=-3C \Rightarrow \quad C=\frac{1}{3}$

$x=0 \quad 1=\frac{8}{9}+-2B+-\frac{1}{3} \Rightarrow B=-\frac{2}{9}$

$\frac{x+1}{(x-1)(x+2)^2} \equiv \frac{2}{9(x-1)}-\frac{2}{9(x+2)}+\frac{1}{3(x+2)^2}$

20. $\frac{3}{x^2(x+3)} \equiv \frac{A}{x+3}+\frac{B}{x}+\frac{C}{x^2}$

$3=Ax^2+B(x+3)x+C(x+3)$

$x=0 \quad 3=3C \quad \Rightarrow \quad C=1$

$x=-3 \quad 3=9A \quad \Rightarrow \quad A=\frac{1}{3}$

$x=1 \quad 3=\frac{1}{3}+4B+4 \Rightarrow B=-\frac{1}{3}$

$\frac{3}{x^2(x+3)} \equiv \frac{1}{3(x+3)}-\frac{1}{3x}+\frac{1}{x^2}$

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21. $\frac{4x}{(x-3)(2x+1)} \equiv \frac{A}{x-3}+\frac{B}{2x+1}$

$4x=\text{A}(2x+1)+\text{B}(x-3)$

$x=3,\ 12=7\text{A} \Rightarrow \text{A}=\frac{12}{7}$

$x=-\frac{1}{2},\ -2=-\frac{7}{2}\text{B} \Rightarrow \text{B}=\frac{4}{7}$

$\frac{4x}{(x-3)(2x+1)} \equiv \frac{12}{7(x-3)}+\frac{4}{7(2x+1)}$

22.

$\frac{x^2+2}{(x+2)(x^2-1)} \equiv \frac{x^2+2}{(x+2)(x+1)(x-1)}=\frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{x-1}$

$4x = A(x+1)(x-1)+B(x+2)(x-1)+C(x+2)(x+1)$

$x=1 \quad 3=6C \quad \Rightarrow \quad c=\frac{1}{2}$

$x=-1 \quad 3=-2B \quad \Rightarrow \quad B=-\frac{3}{2}$

$x=-2 \quad 6=3A \quad \Rightarrow \quad A=2$

$\frac{x^2+2}{(x+2)(x^2-1)} \equiv \frac{2}{x+2}-\frac{3}{2(x+1)}+\frac{1}{2(x-1)}$

23. $\frac{x}{(x+2)(x+1)} \equiv \frac{A}{x+2}+\frac{B}{x+1}$

$x=\text{A}(x+1)+\text{B}(x+2)$

$x=-1 \quad -1=\text{B}$

$x=-2 \quad -2=-\text{A} \quad \Rightarrow \text{A}=2$

$\frac{x}{(x+2)(x+1)}=\frac{2}{x+2}-\frac{1}{x+1}$

$=2(2+x)^{-1}-(1+x)^{-1}=2\times2^{-1}(1+\frac{x}{2})^{-1}-(1+x)^{-1}$

$=1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots-1+x-x^2+x^3-\ldots.$

$=\frac{1}{2}x-\frac{3}{4}x^2+\frac{7}{8}x^3-\ldots.$

Valid for $-1<{}^{x}/_{2}<1$ and $-1<-x<1$

i.e. valid for $-1<x<1$

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24. $\frac{2x}{(2x+1)(x-2)} \equiv \frac{A}{2x+1}+\frac{B}{x-2}$

$2x \equiv \text{A}(x-2)+\text{B}(2x+1)$

$x=2 \quad 4=5\text{B} \Rightarrow \text{B}={}^{4}/_{5}$

$x=-½ \quad -1=-{}^{5}/_{2}\ \text{A} \Rightarrow \quad \text{A}={}^{2}/_{5}$

$\text{f}(x)=\frac{2x}{(2x+1)(x-2)} \equiv \frac{2}{5(2x+1)}+\frac{4}{5(x-2)}$

$={}^{2}/_{5}\,(2x+1)^{-1}+{}^{4}/_{5}\,(x-2)^{-1}$

$\text{f}'(x)=\frac{2}{5}(-1)(2x+1)^{-2}(2)+\frac{4}{5}(-1)(x-2)^{-2}$

$=-\frac{4}{5(2x+1)^2}-\frac{4}{5(x-2)^2}$

25. (a) $\text{f}(-2)=2(-2)^3-3(-2)^2-11(-2)+6$

$=-16-12+22+6=0$

(b) $\text{f}(½)=2(½)^3-3(½)^2-11(½)+6$

$=1/4-3/4-22/4+24/4=0$

$\therefore$ $(2x-1)$ is a factor of $\text{f}(x)$

(c) $\text{f}(x)=(x+2)(2x-1)(x-3)$

26. Let $\text{f}(x)=3x^3+4x^2-3x-4$

$\text{f}(1)=3+4-3-4=0$

$\text{f}(x)=(x-1)(3x^2+7x+4)$

$=(x-1)(3x+4)(x+1)=0$

$x=1\,,-1$ or $-{}^{4}/_{3}$

27. $9x^2+1 \equiv \text{A}x(3x-1)+\text{B}(3x-1)+\text{C}$

$x={}^{1}/_{3} \quad 2=\text{C}$

$x=0 \quad 1=-\text{B}+2 \quad \Rightarrow \text{B}=1$

$x=1 \quad 10=2\text{A}+2+2 \Rightarrow 6=2\text{A} \Rightarrow \text{A}=3$

$\frac{9x^3+1}{3x-1} \equiv 3x+1+\frac{2}{3x-1}$

28. (a) Let $\text{f}(x)=3x^3-2x^2-3x+2$

$\text{f}(-1)=3(-1)^3-2(-1)^2-3(-1)+2$

$=-3-2+3+2=0$

(b) $\text{f}(x)=(x+1)(3x^2-5x+2)$

$=(x+1)(3x-2)(x-1)$

$\frac{(x+1)(3x-2)(x-1)}{(3x-2)(2x+1)}=\frac{(x+1)(x-1)}{(2x+1)}$ or $\frac{x^2-1}{(2x+1)}$

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29. $\frac{3(2+x)-(1-2x)}{(1-2x)(2+x)} \equiv \frac{5+5x}{(1-2x)(2+x)}$ *or* $\frac{5(1+x)}{(1-2x)(2+x)}$

30. $\frac{2-3x}{(1-2x)(1-x)(2+x)} \equiv \frac{A}{1-2x}+\frac{B}{1-x}+\frac{C}{2+x}$

$2-3x=A(1-x)(2+x)+B(1-2x)(2+x)+C(1-2x)(1-x)$

$x=1 \quad -1=-3B \quad \Rightarrow \quad B=\frac{1}{3}$

$x=-2 \quad 8=15C \quad \Rightarrow \quad C=\frac{8}{15}$

$x=\frac{1}{2} \quad \frac{1}{2}=\frac{5}{4}A \quad \Rightarrow \quad A=\frac{2}{5}$

$\frac{2-3x}{(1-2x)(1-x)(2+x)} \equiv \frac{2}{5(1-2x)}+\frac{1}{3(1-x)}+\frac{8}{15(2+x)}$

31. (a) $\frac{5}{x(x+1)^2} \equiv \frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}$

$\equiv \frac{A(x+1)^2+Bx(x+1)+Cx}{x(x+1)^2}$

$5=A(x+1)^2+Bx(x+1)+Cx$

$x=0 \quad 5=A$

$x=-1 \quad 5=-C \Rightarrow \quad C=-5$

$x=1 \quad 5=20+2B-5 \Rightarrow \quad B=-5$

$\frac{5}{x(x+1)^2} \equiv \frac{5}{x}-\frac{5}{(x+1)}-\frac{5}{(x+1)^2}$

$\text{f}(x)=5x^{-1}-5(x+1)^{-1}-5(x+1)^{-2}$

(b) $\text{f}'(x)=-5x^{-2}+5(x+1)^{-2}+10(x+1)^{-3}$

$=-\frac{5}{x^2}+\frac{5}{(x+1)^2}+\frac{10}{(x+1)^3}$

32. $\text{f}(x)=(2x+1)^{-1}+2(x-1)^{-1}$

$=(1+(2x))^{-1}+2(-1)^{-1}(1+(-x))^{-1}$

$=1+(-1)(2x)+\frac{(-1)(-2)}{2\times1}(2x)^2+$

$\frac{(-1)(-2)(-3)}{3\times2\times1}(2x)^3+\ldots..-2\left[1+(-1)(-x)+\right.$

$\left.\frac{(-1)(-2)}{2\times1}(-x)^2+\frac{(-1)(-2)(-3)}{3\times2\times1}(-x)^3+\ldots\right]$

$=-1-4x+2x^2-10x^3+\ldots..$

Valid for $-1<2x<1$ and $-1<-x<1$

i.e. Valid for $-1<{}^{x}/_{2}<1$

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33. Find $\frac{dV}{dt}$ given $\frac{dx}{dt} = 0.3$

$V = x^3$ $\frac{dV}{dx} = 3x^2$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 3x^2 \times 0.3$

$x = 3.5$ $\frac{dV}{dt} = 3 \times 3.5 \times 3.5 \times 0.3 = 11.025$

34. *Want* $\frac{dr}{dt}$ $\frac{dA}{dt} = -20$

$A = \pi r^2$ $\frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{2\pi r} \times (-20) = -\frac{10}{\pi r}$

$10000 = \pi r^2$ $r = 56.418958..$

$\frac{dr}{dt} = -\frac{10}{177.245..} = -0.056$

rate of decrease $0.056\, cm\, s^{-1}$

35. *Want* $\frac{dV}{dt}$ *when* $r = 4$ $\frac{dS}{dt} = 3$

$V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$ $S = 4\pi r^2$ $\frac{dS}{dr} = 8\pi r$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ *But do not know* $\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{dr}{ds} \times \frac{dS}{dt} = \frac{1}{8\pi r} \times 3 = \frac{3}{8\pi r}$

$\frac{dV}{dt} = 4\pi r^2 \times \frac{3}{8\pi r} = \frac{3r}{2}$, $r = 4$, $\frac{dV}{dt} = 6\ cm^3\, s^{-1}$

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36. $\frac{dn}{dt} \propto n$ $\therefore$ $\frac{dn}{dt} = kn$

37. $-\frac{dv}{dt} \propto v$ $\therefore$ $\frac{dv}{dt} = -kv$

38. $\frac{d(y^4)}{dx} = 4y^3 \frac{dy}{dx}$ 39. $\frac{d(\ln y)}{dx} = \frac{1}{y} \times \frac{dy}{dx}$

40. $\frac{d(e^{-y})}{dx} = -e^{-y} \times \frac{dy}{dx}$

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41. $\frac{d(5xy^3)}{dx} = 5x3y^2 \frac{dy}{dx} + 5y^3 = 15xy^2 \frac{dy}{dx} + 5y^3$

42. $\frac{d(3\cos y)}{dx} = -3\sin y \frac{dy}{dx}$

43. $\frac{d(x\sin 2y)}{dx} = 2x\cos 2y \frac{dy}{dx} + \sin 2y$

44. $\frac{d(x^2 \tan y)}{dx} = x^2 \sec^2 y \frac{dy}{dx} + 2x \tan y$

45. $\frac{d(x \ln y)}{dx} = x \frac{1}{y} \frac{dy}{dx} + \ln y$

46. $\frac{d(x^2)}{dx} = 2x$ $\frac{d(y^2)}{dx} = 2y \frac{dy}{dx}$ $\frac{d(25)}{dx} = 0$

$2x + 2y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$

47. $\frac{d(2x^2)}{dx} = 4x$ $\frac{d(xy)}{dx} = x\frac{dy}{dx} + y$ $\frac{d(y)}{dx} = \frac{dy}{dx}$ $\frac{d(2)}{dx} = 0$

$4x + x\frac{dy}{dx} + y - \frac{dy}{dx} = 0$

$x\frac{dy}{dx} - \frac{dy}{dx} = -4x - y$

$\frac{dy}{dx}(x - 1) = -4x - y$

$\frac{dy}{dx} = \frac{-4x-y}{x-1} = \frac{4x+y}{1-x}$

48. $\frac{dy}{dx} = x^2 e^y \frac{dy}{dx} + e^y 2x$

$\frac{dy}{dx} - x^2 e^y \frac{dy}{dx} = e^y 2x$

$\frac{dy}{dx}(1 - x^2 e^y) = 2xe^y$

$\frac{dy}{dx} = \frac{2xe^y}{1-x^2e^y}$

49. $x^{-1} + y^{-1} = \ln y$

$-x^{-2} + -y^{-2} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}$

$\frac{1}{x^2} = \frac{1}{y}\frac{dy}{dx} + \frac{1}{y^2}\frac{dy}{dx}$,

$\frac{dy}{dx} = \frac{-\frac{1}{x^2}}{\frac{1}{y} + \frac{1}{y^2}} = \frac{-y^2}{x^2y + x^2}$

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50. $2x - 6y \frac{dy}{dx} = 4\frac{dy}{dx}$

$2x = \frac{dy}{dx}(4 + 6y)$ $\therefore$ $\frac{dy}{dx} = \frac{2x}{4+6y}$

at (8,4) $\frac{dy}{dx} = \frac{16}{28} = \frac{4}{7}$

$y - 4 = \frac{4}{7}(x - 8)$

$7y - 28 = 4x - 32$ $\therefore$ $7y = 4x - 4$

51. $2y \frac{dy}{dx} = 4$ $\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y} = \frac{2}{\pm\sqrt{4x+1}}$

52. $2x + 2y\frac{dy}{dx} + 4 + 6\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-4-2x}{2y+6}$

At(1, –1) $\frac{dy}{dx} = \frac{-6}{4} = -\frac{3}{2}$

$y - -1 = -\frac{3}{2}(x - 1)$ $\therefore$ $2y + 2 = -3x + 3$

$2y + 3x = 1$

53. $\frac{dy}{dx} = 4^{3x} \ln 4 \times 3 = 3 \times 4^{3x} \ln 4$

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54. (a) $2^x \ln 2$ (b) $3^{2x+1} \ln 3 \times 2$

(c) $5^{x^2} \ln 5 \times 2x$ (d) $-6^{-x} \ln 6$

55. (i) $t = 0$ $5 = A$ $\therefore n = 5 \times 2^t$

$t = 10$ $n = 5 \times 2^{10} = 5120$

(ii) $\frac{dn}{dt} = 5 \times 2^t \ln 2$

$t = 10$, $\frac{dn}{dt} = 5 \times 2^{10} \ln 2 = 3549$ bacteria per min

56. $0.8\, m_0 = m_0 e^{-6k}$

$0.8 = e^{-6k}$

$-6k = \ln 0.8$ $k = 0.03719$

$m = m_0 e^{-0.03719t}$

$t = 10$ $m = 0.689m_0$

$m = 0.5m_0$ $0.5 = e^{-0.3719t}$

$-0.03719t = \ln 0.5$

$t = 18.6$ years (to 3 s.f.)

57. (a) $t = 0$, $P = 26500$

(b) $\frac{dP}{dt} = 26500 \times 1.04^t \ln 1.04$

58. Value, V = £8000 × 0.8^t

After 5 years, V = £8000 × 0.8^5

= £2621.44

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59. $\frac{dy}{dx} = 3x^2 + 6x - 1$, $\frac{d^2y}{dx^2} = 6x + 6$

For point of infl. $6x + 6 = 0$ $\therefore x = -1$

$x = -1, y = -1 + 3 + 1 - 4 = -1$ $\frac{dy}{dx} = -4$ i.e. $\neq 0$

Just before $x = -1$ say $x = -2$ $\frac{d^2y}{dx^2} = -6$ i.e. –ve

Just after $x = -1$ say $x = 0$ $\frac{d^2y}{dx^2} = 6$ i.e. +ve

$\frac{d^2y}{dx^2}$ changes sign $\frac{dy}{dx} \neq 0$

∴ (–1, –1) is a non-stationary point of inflection.

60. $\frac{dy}{dx} = x e^x + e^x$

$\frac{d^2y}{dx^2} = x e^x + e^x + e^x = x e^x + 2e^x$

For point of infl. $x e^x + 2e^x = 0$

$e^x (x + 2) = 0$

$\therefore x = -2$ e^x cannot equal 0

$x = -2, y = -2e^{-2}$, $\frac{dy}{dx} = -2e^{-2} + e^{-2}$

$= -e^{-2}$ i.e. $\neq 0$

Just before $x = -2$ say $x = -3$ $\frac{d^2y}{dx^2} = -e^{-3}$ i.e. –ve

Just after $x = -2$ say $x = -1$ $\frac{d^2y}{dx^2} = e^{-1}$ i.e. +ve

$\frac{d^2y}{dx^2}$ changes sign $\frac{dy}{dx} \neq 0$

∴ (-2, $-\frac{2}{e^2}$) is a non-stationary point of inflection.

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61. Equation of cubic $y = ax^3 + bx^2 + cx + d$

Curve passes through origin $\therefore d = 0$

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

$\frac{d^2y}{dx^2} = 6ax + 2b$ *At* $x = 2$, $\frac{d^2y}{dx^2} = 0$

$\therefore 12a + 2b = 0 \Rightarrow 6a + b = 0$ (i)

$\frac{dy}{dx}$ at origin = –7 $\therefore c = -7$

Equation of cubic $y = ax^3 + bx^2 - 7x$

Passes through (2, – 30)

$\therefore -30 = 8a + 4b - 14 \Rightarrow 2a + b = -4$ (ii)

Solving (i) and (ii)

$4a = 4$ $\therefore a = 1$, $b = -6$

Equation of cubic $y = x^3 - 6x^2 - 7x$

62. Volume of water V = $\frac{1}{3}\pi r^2 h$

Two variables *r* and *h* - need $\frac{dV}{dh}$

tan α = 5/20 =1/4 = *r*/*h*

$r = h/4$

$\therefore V = \frac{1}{3}\pi \frac{h^2}{16} h = \pi \frac{h^3}{48}$

$\frac{dV}{dh} = \pi \frac{h^2}{16}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{16}{\pi h^2} \times 0.2 = \frac{3.2}{\pi h^2}$

$h = 10$, $\frac{dh}{dt} = \frac{3.2}{100\pi} = 0.01$ cm/min to 2 dec. pl.

63. (a) 1 (b) $\frac{dy}{dx}$ (c) $3x^2$ (d) $3y^2 \frac{dy}{dx}$

(e) $15xy^2 \frac{dy}{dx} + 5y^3$ (f) $3x^3 \frac{dy}{dx} + 9x^2 y$

64. (a) $-6\sin 3x$ (b) $-6\sin 3y \frac{dy}{dx}$ (c) $6e^{3x}$

(d) $6e^{3y} \frac{dy}{dx}$ (e) $2\frac{1}{3y} 3\frac{dy}{dx} = \frac{2}{y}\frac{dy}{dx}$

65. $2x + 4y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$ At (5,2) $\frac{dy}{dx} = -\frac{5}{4}$

Equation of tangent $\frac{y-2}{x-5} = -\frac{5}{4}$

$4y - 8 = -5x + 25$

$4y + 5x = 33$

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66. $\frac{dy}{dx} = 2^{3x} \times \ln 2 \times 3 = 3\ln 2 \times 2^{3x}$

67. $10x - (3x\frac{dy}{dx} + y\,3) - 6 \times 2y\frac{dy}{dx} = 0$

$10x - 3x\frac{dy}{dx} - 3y - 12y\frac{dy}{dx} = 0$

$10x - 3y = 3x\frac{dy}{dx} + 12y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{10x-3y}{3x+12y}$

68. (a) $A = 6$

(b) $24 = 6e^{2k}$

$\ln 4 = 2k \Rightarrow k = \frac{\ln 4}{2}$

(c) $N = 6e^{6\ln 4} = 6(e^{\ln 4})^6 = 6 \times 4^6 = 24576$

(d) $\frac{dN}{dt} = 6\frac{\ln 4}{2} e^{t \times \ln 4 / 2} = 3\ln 4\, e^{3\ln 4}$

$= 266$ to 3 sig.fig.

(e) Does not allow for any deaths.
There might not be enough food to support higher numbers of rodents.

69. $\frac{dy}{dx} = 3x^2 + 18x + 15$, $\frac{d^2y}{dx^2} = 6x + 18$

For point of infl. $6x + 18 = 0$ $\therefore x = -3$

$x = -3, y = -27 + 81 - 45 - 4 = 5$

$\frac{dy}{dx} = 27 - 54 + 15 = -12$ i.e. $\neq 0$

Just before $x = -3$ say $x = -4$ $\frac{d^2y}{dx^2} = -6$ i.e. –ve

Just after $x = -3$ say $x = -2$ $\frac{d^2y}{dx^2} = 6$ i.e. +ve

$\frac{d^2y}{dx^2}$ changes sign $\frac{dy}{dx} \neq 0$

∴ (– 3, 5) is a non-stationary point of inflection.

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70. (a) $\delta x = 0.2$

(b) $x = 1 \quad y = 3 \quad y\delta x = 0.6$

$x = 1.2 \quad y = 3.4224 \quad y\delta x = 0.684489$

$x = 1.4 \quad y = 4.25420 \quad y\delta x = 0.850841$

$x = 1.6 \quad y = 5.486625 \quad y\delta x = 1.097325$

$x = 1.8 \quad y = 7.140642 \quad y\delta x = 1.428128$

Estimated area under graph = 4.7 sq units to 1 d.pl.

Note: A better estimate can be made by using more bars with smaller width.

(c) $\int_1^2 x^{-2} + x^3 + 1\ dx = \left[\frac{x^{-1}}{-1} + \frac{x^4}{4} + x\right]_1^2$

$= (-{}^1\!/_2 + 4 + 2) - (-1 + ¼ + 1)$

$= 5.25$

71. $\int x^3 + 3x^{1/2} - 5\ dx = \frac{x^4}{4} + \frac{3x^{3/2}}{3/2} - 5x + c$

$= \frac{x^4}{4} + 2x^{3/2} - 5x + c$

72. $\int 2x^{-2} - 3x^{-3}\ dx = \frac{2x^{-1}}{-1} - \frac{3x^{-2}}{-2} + c$

$= -\frac{2}{x} + \frac{3}{2x^2} + c$

73. $\frac{(5x-2)^5}{5\times5} + c = \frac{(5x-2)^5}{25} + c$

74. $\frac{2(2x+1)^5}{6\times2} + c = \frac{(2x+1)^5}{6} + c$

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75. $\int 2(2x+1)^{-3} dx = \frac{2(2x+1)^{-2}}{-2\times2} + c$

$= -\frac{1}{2(2x+1)^2} + c$

76. $\int \frac{2}{5}(3x-1)^{-2} dx = \frac{2(3x-1)^{-1}}{5\times(-1)\times3} + c$

$= -\frac{2}{15(3x-1)} + c$

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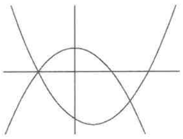
77. $x^2 - 3x - 18 = 9 - x^2$

$2x^2 - 3x - 27 = 0$

$(2x-9)(x+3) = 0$

Intersect at $x = -3$ or $9/2$



$$\text{Area} = \int_{-3}^{9/2} 9 - x^2 - (x^2 - 3x - 18)\,dx$$

$$= \int_{-3}^{9/2} -2x^2 + 3x + 27\,dx$$

$$= \left[-\frac{2x^3}{3} + \frac{3x^2}{2} + 27x\right]_{-3}^{9/2}$$

$$= \left[-\frac{243}{4} + \frac{243}{8} + \frac{243}{2}\right] - \left[2\times9 + \frac{27}{2} - 81\right]$$

$= \frac{729}{8} + \frac{99}{2} = \frac{1125}{8} = 140.625$ sq. units.

78. MEI students

$\int_0^3 3y - y^2\ dy = \left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$

$= \left[\frac{27}{2} - \frac{27}{3}\right] - [0] = 4.5$ sq. units.

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79. $\int 3e^x dx = 3e^x + c$

80. $\int \frac{1}{2x} dx = \frac{1}{2}\ln|x| + c$

81. $\int \frac{5}{7x} dx = \frac{5}{7}\ln|x| + c$

82. $\int \frac{1}{3}e^x dx = \frac{1}{.3}e^x + c$

83. $\int 5x^{-1} dx = \int \frac{5}{x} dx = 5\ln|x| + c$

84. $\int 5 - \frac{2}{3x} dx = 5x - \frac{2}{3}\ln|x| + c$

85. $\int 5(e^x + 3)dx = \int 5e^x + 15dx = 5e^x + 15x + c$

86. $\int (1 - \frac{1}{x})^2 dx = \int 1 - \frac{2}{x} + \frac{1}{x^2} dx = x - 2\ln|x| + \frac{x^{-1}}{-1}$

$= x - 2\ln|x| - \frac{1}{x} + c$

87. $3\ln|x+2|$

88. $\frac{1}{2}\ln|2x+3|$

89. $\frac{3}{5}\ln|5x+3|$

90. $\frac{4}{-2}\ln|5-2x| = -2\ln|5-2x|$

91. $\int e^x - \sqrt{x}\,dx = \int e^x - x^{\frac{1}{2}} dx$

$$= e^x - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = e^x - \frac{2x^{\frac{3}{2}}}{3} + c$$

92. $\int \frac{4}{5}x^{-1} + \frac{3}{2}x^{-2}\,dx = \frac{4}{5}\ln|x| + \frac{3}{2}\frac{1}{-1}x^{-1} + c$

$= \frac{4}{5}\ln|x| - \frac{3}{2x} + c$

93. $\int_{-1}^{1} 2e^x + 2dx = \left[2e^x + 2x\right]_{-1}^{1}$

$= (2e + 2) - (\frac{2}{e} - 2) = 2e - \frac{2}{e} + 4$

94. $\int_1^2 \frac{3}{2x} dx = \left[\frac{3}{2}\ln x\right]_1^2 = \frac{3}{2}\ln 2 - \frac{3}{2}\ln 1 = \frac{3}{2}\ln 2$

95. $A = \int_1^2 x^2 + e^x dx = \left[\frac{x^3}{3} + e^x\right]_1^2 = \left(\frac{8}{3} + e^2\right) - \left(\frac{1}{3} + e\right)$

$= \frac{7}{3} + e^2 - e = 7.00$ *square units to 3 s.f.*

Page24

96. $-\frac{\cos 3x}{3} + c$

97. $\frac{2\sin 6x}{6} + c = \frac{1}{3}\sin 6x + c$

98. $-\cos(x - \frac{\pi}{4}) + c$

99. $\frac{3\sin(\pi - 2x)}{-2} + c = -\frac{3}{2}\sin(\pi - 2x) + c$

100. $\left[\frac{-2\cos 2x}{2}\right]_0^{\frac{\pi}{6}} = -\frac{1}{2} - (-1) = \frac{1}{2}$

101. $\left[\frac{3\sin(2x + \frac{\pi}{2})}{2}\right]_{\pi/6}^{\pi/2} = \left(-\frac{3}{2}\right) - \left(\frac{3}{2}\frac{1}{2}\right) = -\frac{9}{4}$

102. $2\tan x + c$

103. $\frac{1}{3}\tan 3x + c$

104. $\int \sin^2 x dx = \frac{1}{2}\int 1 - \cos 2x\,dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

105. $\int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + c$

$(= \ln|\sec x| + c)$

106. $\int \sec^2 x\,dx = \tan x + c$

107. $\cos 2x = 2\cos^2 x - 1$

$4\cos^2 x = 2\cos 2x + 2$

$\int_0^{\pi/2} 2\cos 2x + 2\,dx = \left[2\frac{\sin 2x}{2} + 2x\right]_0^{\pi/2}$

$= (0 + \pi) - (0 + 0) = \pi$

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108. $\cos 4\theta = 2\cos^2 2\theta - 1$

$\cos^2 2\theta = ½\cos 4\theta + ½$

$\int_0^{\pi/4} \cos^2 2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}\cos 4\theta + \frac{1}{2}d\theta$

$= \left[\frac{1}{2}\frac{\sin 4\theta}{4} + \frac{1}{2}\theta\right]_0^{\frac{\pi}{4}}$

$= \left[0 + \frac{\pi}{8}\right] - [0 + 0] = \frac{\pi}{8}$

109. $\frac{du}{dx} = 2x \quad \therefore \quad du = 2x\ \mathrm{dx}$

$\int x(x^2+5)^5 dx = \frac{1}{2}\int u^5 du$

$= \frac{1}{2}\frac{1}{6}u^6 + c$

$= \frac{1}{12}(x^2 + 5)^6 + c$

110. $\frac{du}{dx} = 2x \quad \therefore \quad du = 2x\ \mathrm{dx}$

$\int x e^{x^2} dx = \frac{1}{2}\int e^u du$

$\frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c$

111. $u = \sin x \quad du = \cos x\ \mathrm{dx}$

$\int e^u du = e^u + c = e^{\sin x} + c$

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112. $u = \sin x \quad du = \cos x\ \mathrm{dx}$

$\int u^3 du = \frac{u^4}{4} + c = \frac{1}{4}\sin^4 x + c$

113. $u = \cos x \quad du = -\sin x\ \mathrm{dx} \quad -du = \sin x\ \mathrm{dx}$

$x = \pi/2 \quad u = 0 \qquad x = 0 \quad u = 1$

$\int_1^0 -u^4 du = \left[-\frac{u^5}{5}\right]_1^0 = 0 - -\frac{1}{5} = \frac{1}{5}$

114. $\ln|x^2 + 4| + c$

115. the top is the differential of the bottom

$\ln|1 - \cos x| + c$

116. $\frac{1}{3}\int \frac{3x^2}{x^3 + 4}dx = \frac{1}{3}\ln|x^3 + 4| + c$

117.

$3\int \frac{x}{x^2+1}dx = \frac{3}{2}\int \frac{2x}{x^2+1}dx = \frac{3}{2}\ln|x^2+1| + c$

118. $\frac{1}{2}\int \frac{2x+1}{x^2+2}dx = \frac{1}{2}\ln|x^2 + 2x| + c$

119. $\frac{2}{3}\int \frac{3x^2}{x^3+1}dx = \frac{2}{3}\ln|x^3 + 1| + c$

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120. $u = x \quad \frac{du}{dx} = 1 \quad \frac{dv}{dx} = e^{-x} \quad v = -e^{-x}$

$\int xe^{-x} dx = -xe^{-x} + \int e^{-x}\ dx$

$= -xe^{-x} - e^{-x} + c$

121. $u = x + 1 \quad \frac{du}{dx} = 1 \quad \frac{dv}{dx} = e^x \quad v = e^x$

$\int (x+1)e^x dx = (x+1)e^x - \int e^x\ dx$

$= xe^x + e^x - e^x + c = xe^x + c$

122. $u = 2x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = \cos 3x \quad v = \frac{1}{3}\sin 3x$

$\int 2x\cos 3x\ dx = \frac{2}{3}x\sin 3x - \int \frac{2}{3}\sin 3x\,dx$

$= \frac{2}{3}x\sin 3x + \frac{2}{3}\frac{\cos 3x}{3} + c = \frac{2}{3}x\sin 3x + \frac{2}{9}\cos 3x + c$

123. $u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad \frac{dv}{dx} = 1 \quad v = x$

$\int \ln x\ \ dx = x\ln x - \int \frac{1}{x}x\ dx$

$= x\ln x - \int 1\ dx = x\ln x - x + c$

124. $u = x^2 \quad \frac{du}{dx} = 2x \quad \frac{dv}{dx} = \sin x \quad v = -\cos x$

$\int x^2 \sin x\ dx = x^2\cos x + \int 2x\cos x\ \ dx$

$u = 2x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = \cos x \quad v = \sin x$

$\int x^2\sin x\ dx = -x^2\cos x + \left(2x\sin x - \int 2\sin x dx\right)$

$= -x^2\cos x + 2x\sin x + 2\cos x + c$

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125. $u = x^2 \quad \frac{du}{dx} = 2x \quad \frac{dv}{dx} = e^x \quad v = e^x$

$\int x^2 e^x dx = x^2 e^x - \int 2xe^x\ dx$

$u = 2x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = e^x \quad v = e^x$

$\int x^2 e^x dx = x^2 e^x - \left\{2xe^x - \int 2e^x dx\right\}$

$= x^2 e^x - 2xe^x + 2e^x + c$

126. $u = 2x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = \sin 2x \quad v = -\frac{\cos 2x}{2}$

$\int_0^{\pi/2} 2x\sin 2x\ dx = -x\cos 2x - \int -\cos 2x\ dx$

$= \left[-x\cos 2x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \left(-\frac{\pi}{2}(-1) + 0 - (0)\right) = \frac{\pi}{2}$

127. $u = 10x \quad \frac{du}{dx} = 10 \qquad \frac{dv}{dx} = e^{5x} \quad v = \frac{e^{5x}}{5}$

$$\int_0^1 10\,xe^{x}dx = \tfrac{10}{5}xe^{5x} - \int \tfrac{10}{5}e^{5x}\,dx$$

$$= \left[2xe^{5x} - \tfrac{2}{5}e^{5x}\right]_0^1$$

$$= \left(2e^5 - \tfrac{2}{5}e^5\right) - \left(0 - \tfrac{2}{5}\right) = \tfrac{8}{5}e^5 + \tfrac{2}{5}$$

128. $u = e^x \quad \frac{du}{dx} = e^x \qquad \frac{dv}{dx} = \sin x \quad v = -\cos x$

$$\int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx$$

$$u = e^x \quad \frac{du}{dx} = e^x \qquad \frac{dv}{dx} = \cos x \qquad v = \sin x$$

$$\int e^x \sin x\,dx = -e^x\cos x + \left(e^x \sin x - \int e^x \sin x dx\right)$$

$$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x$$

$$\int e^x \sin x\,dx = -\frac{1}{2}e^x\cos x + \frac{1}{2}e^x \sin x + c$$

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129. $\frac{2x}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$

$2x = A(x-3) + B(x+2)$

$x = -2 \qquad -4 = -5A \quad \Rightarrow \quad A = \frac{4}{5}$

$x = 3 \qquad 6 = 5B \quad \Rightarrow \quad B = \frac{6}{5}$

$$\frac{2x}{(x+2)(x-3)} = \frac{4}{5(x+2)} + \frac{6}{5(x-3)}$$

$$\int \frac{2x}{(x+2)(x-3)}dx = \int \frac{4}{5(x+2)} + \frac{6}{5(x-3)}dx$$

$$= \frac{4}{5}\ln|x+2| + \frac{6}{5}\ln|x-3| + \ln A = \ln A(x+2)^{4/5}(x-3)^{6/5}$$

130. $\frac{2x-5}{(x+2)(x-1)^2} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$

$2x - 5 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

$x = 1 \quad -3 = 3C \quad \Rightarrow \quad C = -1$

$x = -2 \quad -3 = 9A \quad \Rightarrow \quad A = -1$

$x = 0 \qquad -5 = -1 - 2B - 2 \Rightarrow B = 1$

$$\frac{2x-5}{(x+2)(x-1)^2} = \frac{-1}{(x+2)} + \frac{1}{(x-1)} - \frac{1}{(x-1)^2}$$

$$\int \frac{2x-5}{(x+2)(x-1)^2}dx = \int \frac{-1}{(x+2)} + \frac{1}{(x-1)} - \frac{1}{(x-1)^2}dx$$

$$= -\ln|x+2| + \ln|x-1| - \frac{(x-1)^{-1}}{-1} + c$$

$$= \ln\left|\frac{x-1}{x+2}\right| + \frac{1}{x-1} + c$$

131. $6x^3/3 + 2x^2/2 - x + c = 2x^3 + x^2 - x + c$

132. $5e^{2x}/2 + c$

133. $\frac{3}{2}\ln|2x| + c$ or $\frac{3}{2}\ln|x| + c$

134. $\frac{2}{3}\ln|3x-2| + c$

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135. $\frac{2\ln|1-x|}{-1} + c = -2\ln|1-x| + c$

136. $\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}\times 2} + c = \frac{(2x+1)^{\frac{3}{2}}}{3} + c$

137. $\int 2(1-x)^{-4}\,dx = 2\frac{(1-x)^{-3}}{(-3)(-1)} + c$

$$= \frac{2}{3(1-x)^3} + c$$

138. $\int 5(1-2x)^{-2}\,dx = 5\frac{(1-2x)^{-1}}{(-1)(-2)} + c$

$$= \frac{5}{2(1-2x)} + c$$

139. $u = x^2 - 2 \qquad du = 2x\,dx \qquad \frac{du}{2} = xdx$

$$\int u^4\frac{du}{2} = \frac{1}{2}\frac{u^5}{5} + c = \frac{1}{10}\left(x^2-2\right)^5 + c$$

140. $u = 2 + e^x \qquad du = e^x dx$

$$\int u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\left(2+e^x\right)^{\frac{3}{2}} + c$$

141. $\left[\frac{(2x-1)^4}{4\times 2}\right]_1^2 = \frac{81}{8} - \frac{1}{8} = 10$

142. $u = \ln x \qquad du = \frac{1}{x}dx \qquad x = 1 \quad u = \ln 1 = 0$

$x = 2 \quad u = \ln 2$

$$\int_0^{\ln 2} u\ du = \left[\frac{u^2}{2}\right]_0^{\ln 2} = \frac{(\ln 2)^2}{2}$$

143. $u = x \quad \frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \cos x \quad v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\ dx$$

$$= x\sin x - -\cos x + c \ = x\sin x + \cos x + c$$

144. $\frac{5x^2+3}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$5x^2 + 3 = A(x+1)^2 + B(x+1)(x-1) + C(x-1)$

$x = 1 \qquad 8 = 4A \qquad \Rightarrow \quad A = 2$

$x = -1 \qquad 8 = -2C \qquad \Rightarrow \quad C = -4$

$x = 0 \qquad 3 = 2 + -B + 4 \quad \Rightarrow \quad B = 3$

so $\frac{5x^2+3}{(x-1)(x+1)^2} = \frac{2}{x-1} + \frac{3}{x+1} - \frac{4}{(x+1)^2}$

$$\int \frac{5x^2+3}{(x-1)(x+1)^2}dx = \int \frac{2}{x-1} + \frac{3}{x+1} + \frac{4}{(x+1)^2}\ dx$$

$$= 2\ln(x-1) + 3\ln(x+1) - \frac{4(x+1)^{-1}}{-1} + c$$

$$= 2\ln(x-1) + 3\ln(x+1) + \frac{4}{x+1} + c$$

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145. $u = x^3 + 3 \qquad du = 3x^2\,dx$

$x = 1 \quad u = 4 \quad$ and $\quad x = 2 \quad u = 11$

$$\int_1^2 \frac{x^2}{x^3+3}dx = \int_4^{11}\frac{1}{u}\frac{du}{3} = \frac{1}{3}\int_4^{11}\frac{1}{u}du$$

$$= \frac{1}{3}\left[\ln u\right]_4^{11} = \frac{1}{3}\left(\ln 11 - \ln 4\right)$$

$$= \frac{1}{3}\ln\frac{11}{4}$$

146. $x = 1 \quad y = \log 1 = 0 \quad x = 1.5 \quad y = \log 1.5$

$x = 2 \quad y = \log 2$

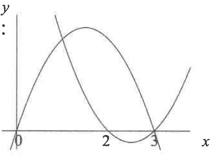
$$\int_1^2 \log x\,dx \approx \frac{0.5}{2}\left[0 + 2\times\log 1.5 + \log 2\right]$$

$$\approx \frac{1}{4}\log\frac{9}{2} = 0.163 \textit{ to } 3\ s.f.$$

Under-estimate because the trapeziums are below the curve $y = \log_{10}x$

147. $\int_1^2 (3x-5)^4\,dx = \left[\frac{(3x-5)^5}{5\times 3}\right]_1^4 = \frac{1}{15} - \frac{-32}{3} = \frac{33}{15}$

148.



Intersections with x axis:

$x(3-x) = 0$

$x = 0$ or 3

$(x-3)(x-2) = 0$

$x = 2$ or 3

Intersection of curves

$x^2 - 5x + 6 = 3x - x^2$

$2x^2 - 8x + 6 = 0$

$(2x-2)\,(x-3) = 0 \quad \Rightarrow \quad x = 1$ or 3

$$A = \int_1^3 3x - x^2 - (x^2 - 5x + 6)dx = \int_1^3 -2x^2 + 8x - 6)dx$$

$$= \left[-\frac{2x^3}{3} + 4x^2 - 6x\right]_1^3 = 0 - \frac{-8}{3} = \frac{8}{3}\text{ sq. units.}$$

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149. $\int \frac{1}{y}dy = \int 2x\,dx$

$\ln y = x^2 + c$

$y = e^{x^2+c} = e^{x^2}e^c = Ae^{x^2}$

150. $\int dy = \int 2x^2 + 5\,dx$

$y = \frac{2}{3}x^3 + 5x + c$

$10 = 18 + 15 + c \Rightarrow c = -23$

$y = \frac{2}{3}x^3 + 5x - 23$

151. $\int \frac{1}{y}dy = \int \frac{1}{x}dx$

$\ln y = \ln x + \ln A \quad \Rightarrow \quad \ln y = \ln Ax$

$y = Ax$

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152. Area, $A = s^2 \qquad \frac{dA}{ds} = 2s$

$\frac{dA}{dt} \propto s \qquad \frac{dA}{dt} = ks$

$$\frac{ds}{dt} = \frac{ds}{dA}\times\frac{dA}{dt} = \frac{1}{2s}\times ks = \frac{k}{2}$$

153. $\frac{dx}{dt} = -kx$

$$\int \frac{1}{x}\,dx = \int -k \quad dt$$

$lnx = -kt + c$

$x = e^{-kt+c} = e^{-kt}\,e^c$

$x = Ae^{-kt}$

154. $\frac{dr}{dt} = kr$

$$\int \frac{1}{r}\,dr = \int k \quad dt$$

$lnr = kt + c$

$r = e^{kt+c} = e^{kt}e^c$

$r = Ae^{kt}$

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155. $\frac{d\theta}{dt} = -k\theta$

$$\int \frac{1}{\theta}\,dx = \int -k \quad dt$$

$ln\,\theta = -kt + c$

$\theta = e^{-kt+c} = e^{-kt}e^c$

$\theta = Ae^{-kt}$

$t = 0,\ \theta = 100 \quad \therefore \quad 100 = A$

$\theta = 100\,e^{-kt}$

$t = 30,\ \theta = 50 \quad \therefore \quad 50 = 100\,e^{-30k}$

$0.5 = e^{-30k}$

$\ln 0.5 = -30k$

$k = \ln 0.5 \div (-30) = 0.023$

$\therefore \quad \theta = 100\,e^{-0.023t}$

156. $\frac{dP}{dt} = kP$

$$\int \frac{dP}{P} = \int kdt$$

$\ln P = kt + c$

$P = e^{kt+c} = e^c e^{kt} = Ae^{kt}$

$t = 0 \quad P = 50000 \quad \therefore \quad A = 50000$

$P = 50000\,e^{kt}$

$t = 50 \quad 10000 = 50000\,e^{50k}$

$2 = e^{50k}$

$\ln 2 = 50k \qquad k = 0.01386$

$t = 70 \quad P = 50000\,e^{0.01386\times 70} = 131951$

$200000 = 50000\,e^{0.01386t}$

$t = \ln 4/0.01386 = 100$ i.e. In year 2100

157. $\frac{dx}{dt} = -kx$

$$\int \frac{dx}{x} = \int -kdt$$

$\ln x = -kt + c$

$x = e^{-kt+c} = e^c e^{-kt} = Ae^{-kt}$

$t = 0 \quad x = 100 \quad \therefore \quad A = 100$

$x = 100\,e^{-kt}$

$t = 20 \quad x = 60 \quad 60 = 100\,e^{-20k}$

$0.6 = e^{-20k}$

$\ln 0.6 = -20k \quad \therefore \quad k = 0.02554$

$x = 100\,e^{-0.02554t}$

(i) $t = 40 \quad P = 100\,e^{-0.02554\times 40} = 36$

(ii) $30 = 100\,e^{-0.02554t}$

$t = \ln 0.3/(-0.02554) = 47.1$ minutes

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158. $t = y/5 \qquad x = 2y^2/25 \qquad 2y^2 = 25x$

159. $x = 4/y \qquad xy = 4$

160. $\left(\frac{x}{3}^2\right) + \left(\frac{y}{3}\right)^2 = 1 \qquad x^2 + y^2 = 9$

161. $\ln x = t \qquad y = \cos(\ln x)$

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162. (a) $\frac{dx}{dt} = 3 \quad \frac{dy}{dt} = 4t \quad \frac{dy}{dx} = 4t\times\frac{1}{3} = \frac{4}{3}t$

(b) $\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 6t^2 \quad \frac{dy}{dx} = 6t^2\times\frac{1}{2t} = 3t$

(c) $\frac{dx}{dt} = -2\sin t \quad \frac{dy}{dt} = 3\cos t \quad \frac{dy}{dx} = 3\cos t\times\frac{1}{-2\sin t} = -\frac{3}{2}\cot t$

(d) $\frac{dx}{dt} = -t^{-2} \quad \frac{dy}{dt} = 4t \quad \frac{dy}{dx} = 4t\times\frac{1}{-1/t^2} = -4t^3$

163. $\frac{dx}{dt}=2t \quad \frac{dy}{dt}=5 \quad \frac{dy}{dx}=5\times\frac{1}{2t}=\frac{5}{2t}$

$t=1 \quad \frac{dy}{dx}=\frac{5}{2}$ *and* $x=1, y=5$

$y-5=\frac{5}{2}(x-1)$

$2y-10=5x-5$

$2y=5x+5$

164. $\frac{dx}{dt}=1 \quad \frac{dy}{dt}3t^2-3 \quad \frac{dy}{dx}=(3t^2-3)\times\frac{1}{1}=3t^2-3$

$0=3t^2-3=3(t^2-1)=3(t+1)(t-1)$

$t=-1 or 1$

$\frac{d^2y}{dx^2}=6t\times\frac{1}{1}=6t$

$t=-1 \quad \frac{d^2y}{dx^2}=-6$ *i.e.* $-ve$ $\therefore$ maximum at $(-1,2)$

$t=1 \quad \frac{d^2y}{dx^2}=6$ *i.e.* $+ve$ $\therefore$ minimum at $(1,-2)$

165. $\left(\frac{x}{2}\right)^2+\left(\frac{y}{5}\right)^2=1$ *or* $25x^2+4y^2=100$

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166. $\frac{dx}{d\theta}=-5\sin\theta \quad \frac{dy}{d\theta}=5\cos\theta$

$\frac{dy}{dx}=5\cos\theta\times\frac{1}{-5\sin\theta}=-\frac{\cos\theta}{\sin\theta}$

gradient of normal $=\frac{\sin\theta}{\cos\theta}=\tan\theta$

at $\theta={}^{\pi}\!/_{3}$ *gradient is* $\tan{}^{\pi}\!/_{3}=\sqrt{3}$

and $x=\frac{5}{2}, y=5\frac{\sqrt{3}}{2}$

$y-5\frac{\sqrt{3}}{2}=\sqrt{3}\left(x-\frac{5}{2}\right)$

$y=\sqrt{3}x$

Cartesian equation $x^2+y^2=25$

167. $\frac{dx}{d\theta}=-4\sin\theta$

$$A=\int_{\pi/2}^{0}2\sin\theta\times(-4\sin\theta)d\theta=\int_{\pi/2}^{0}-8\sin^2\theta\,d\theta$$

$$=\int 4\cos 2\theta-4\,d\theta=[2\sin 2\theta-4\theta]_{\pi/2}^{0}=2\pi$$

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168. $-5t^2+20\sqrt{(2)}\,t+1.8=0$

$t>0$ so $t=5.71979$

$x=(5\cos 45^\circ)\times 5.71979=20.2$ m to 1 d.pl.

169. (a) Hor: velocity $=(-7-8)/5=-3\ \text{ms}^{-1}$

Vert: velocity $=(13-3)/5=2\ \text{ms}^{-1}$

(b) $x=8-3t,\ y=3+2t$

170. $f(x)=x^3-12x^2+48x-61$

$f(0)=-61$

$f(1)=1-12+48-61=-24$

$f(2)=8-48+96-61=-5$

$f(3)=27-108+144-61=2$

Change of sign $\therefore$ root lies between $x=2$ and $x=3$

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171. $f(x)=x^4+x^3-4x^2-5$

$f(-2.7)=-0.70$

$f(-2.8)=3.15$

Change of sign

$\therefore$ root lies between $x=-2.7$ and $x=-2.8$.

172. $x^3+3x^2-7=2x+2$

$x^3+3x^2-2x-9=0$

$f(x)=x^3+3x^2-2x-9$

$f(1.6)=-0.424$

$f(1.7)=1.183$

Change of sign $\therefore$ point of intersection lies between $x=1.6$ and $x=1.7$

173. (i) $e^x=5-x$

$\ln e^x=\ln(5-x)$

$x\ln e=\ln(5-x) \qquad \ln e=1$

$x=\ln(5-x)$

(ii) $x_1=1$

$x_2=\ln(5-1)=1.386$

$x_3=\ln(5-1.386....)=1.285$

$x_4=\ln(5-1.285....)=1.312$

(iii) $f(x)=e^x+x-5$

$f(1.305)=-0.00731 \qquad f(1.315)=0.03975$

Change of sign - therefore a root is 1.31 to 2 d.p.

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174. $x_1=2, \quad x_2=2.333333,$

$x_3=2.290249, \quad x_4=2.289429$

$3x^3=4\times3+2x^3$ (multiplying throughout by $2x^3$)

$x^3-12=0$

175.(a) converges - cobweb (b) diverges - cobweb

(c) converges - staircase (d) diverges - staircase

(i) $g'(x)$ negative - cobweb

$g'(x)$ positive - staircase

(ii) $-1<g'(x)<1$ converges

otherwise does not converge.

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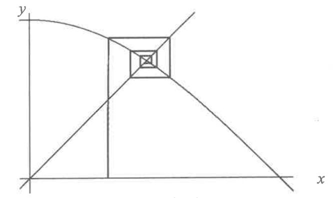
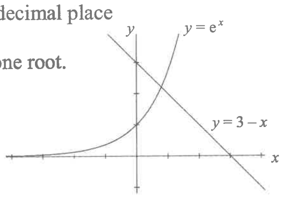
176.



$x=0.7$ to one decimal place

177. One point of intersection so one root.



$f(x)=e^x+x-3$

$f'(x)=e^x+1$

$x_2=0.821392..$

$x_3=0.792355..$

$x_4=0.792059..$

$x_5=0.792059..$

$f(0.7915)=-0.002.. \quad f(0.7925)=0.001..$

Change of sign. $\therefore x=0.792$ to 3 decimal places.

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178. $f(-3)=12 \quad f(-4)=-16$ Change of sign.

$\therefore$ Root between $x=-3$ and $x=-4$

$x_1=-4,\ x_2=-3.58974359,\ x_3=-3.52397306$

$x_4=-3.522334399,\ x_4=-3.5223333$

$f(-3.5215)=0.02.. \quad f(-3.5225)=-0.0047..$

Change of sign. $\therefore x=-3.522$ to 3 decimal places.

179. $f(0.5)=0.0403 \quad f(0.6)=-0.2376$

Change of sign.

$\therefore$ Root between $x=0.5$ and $x=0.6$

$x_1=0.5,\ x_2=0.51502...,\ x_3=0.514933...,$

$x_4=0.514933$

$f(0.5145)=0.001.. \quad f(0.5155)=-0.0015..$

Change of sign. $\therefore x=0.515$ to 3 decimal places.

180. $x_0=1 \quad y_0=\sqrt{(1^2+1)}=1.4142$

$x_1=1.25 \quad y_1=1.6008$

$x_2=1.5 \quad y_2=1.8028$

$x_3=1.75 \quad y_3=2.0156$

$x_4=2 \quad y_4=2.2361 \quad h=¼$

$\int_1^2\sqrt{(x^2+1)}\,dx$

$\approx\frac{1}{8}(1.4142+3.2016+3.6056+4.0312+2.2361)$

$\approx 1.8111=1.81$ to two dec.pl.

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181. $x_0=0 \quad y_0=0$

$x_1=\pi/6 \quad y_1=0.125$

$x_2=\pi/3 \quad y_2=0.649519$

$x_3=\pi/2 \quad y_3=1$

$A=\int_0^{\pi/2}\sin^3x\,dx$

$\approx\pi/12(0+1+2\times0.125+2\times0.649519)$

≈ 0.67 Sq. units to 2 d.pl.

182. $x=1 \quad y=\ln 1=0$

$x=1.5 \quad y=\ln 1.5$

$x=2 \quad y=\ln 2$

$\int_1^2\ln x\,dx\approx\frac{0.5}{2}[0+2\times\ln 1.5+\ln 2]=\frac{1}{4}\ln\frac{9}{2}=0.376$

Under-estimate – the trapezia are below the curve.

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183. $\overrightarrow{AB}=\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$ Distance $=\sqrt{(1+1+4)}=\sqrt{6}$

184. $\overrightarrow{AB}=5\boldsymbol{i}-10\boldsymbol{j}+5\boldsymbol{k} \quad \overrightarrow{AP}=3/5(5\boldsymbol{i}-10\boldsymbol{j}+5\boldsymbol{k})$

$=3\boldsymbol{i}-6\boldsymbol{j}+3\boldsymbol{k}$

$\overrightarrow{OP}=\boldsymbol{i}+6\boldsymbol{j}+3\boldsymbol{k}+3\boldsymbol{i}-6\boldsymbol{j}+3\boldsymbol{k}=4\boldsymbol{i}+6\boldsymbol{k}$

Coordinates of P (4, 0, 6)

185. $\overrightarrow{AB}=4\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k},\ \overrightarrow{AC}=\boldsymbol{i}-6\boldsymbol{k},\ \overrightarrow{BC}=-3\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}$

$AB=\sqrt{21} \quad AC=\sqrt{37} \quad BC=\sqrt{62}$

$\cos(\text{angle ACB})=\frac{37+62-21}{2\sqrt{37}\sqrt{62}}=0.814269$

Angle ACB $=35.5^\circ$

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186. Assume p is not even.

p^2, is even(Given). If p is not even, it is odd, and therefore of the form $2n+1$, where n is a whole number. Then $p^2=(2n+1)^2=4n^2+4n+1$

$=2(2n^2+2n)+1$

So $4n^2+4n$ is clearly even, so $4n^2+4n+1$ is odd. This means p^2 is odd and have a contradiction. Therefore the assumption that p is not even must be wrong, i.e. p is even.

187. *Assume that the sum of a rational number and an irrational number is rational. The first rational number can be expressed as p/q where p and q are coprime integers. Let the irrational number be a Their sum can be expressed as r/s where r and s are coprime integers. Then,*

$$\frac{p}{q}+a=\frac{r}{s}$$

$$a=\frac{r}{s}-\frac{p}{q}$$

$$=\frac{qr-ps}{qs}$$

$qr-ps$ is an integer, and so is qs Therefore, a is a rational number. However, a was defined to be irrational. This is a contradiction. Hence, the sum of a rational number and an irrational number is irrational.

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188. *Since $x\in[0,\pi/2]$, neither sin x nor cos x is negative*

Assume sin x+cos x < 1

So $(\sin x+\cos x)^2<1^2$, which gives

$\sin^2x+2\sin x\cos x+\cos^2x<1^2$

As $\sin^2x+\cos^2x=1$, this becomes

$1+2\sin x\cos x<1$,

Subtracting 1 from both sides gives $2\sin x\cos x<0$. But this contradicts the fact that neither sinx nor cosx is negative.

So $\sin x+\cos x\geq 1$

189. (i) $x=-2/3$

$-9\times 8/27+9\times 4/9+4\times 2/3-4$

$=-8/3+4+8/3-4=0$

$\therefore$ $(3x+2)$ is a factor

(ii) $\frac{(3x+2)(3x-2)(x+1)}{(3x+2)(x+1)}=3x-2$

190. $\frac{2}{3}\int\frac{3x^2}{x^3+1}dx \quad \frac{2}{3}\ln|x^3+1|$

191.

(i) $\frac{d}{dx}(\tan x)=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\times\cos x-\sin x\times(-\sin x)}{\cos^2x}$

$=\frac{\cos^2x+\sin^2x}{\cos^2x}=\frac{1}{\cos^2x}=\sec^2x$

(ii) $u=x \quad \frac{du}{dx}=1 \quad \frac{dv}{dx}=\sec^2x \quad v=\tan x$

$\int x\sec^2x\,dx=x\tan x-\int\tan x\,dx$

$=x\tan x-\int\frac{\sin x}{\cos x}dx=x\tan x+\int\frac{-\sin x}{\cos x}dx$

$=x\tan x+\ln|\cos x|+c$

192. $V=\frac{4}{3}\pi r^3 \quad \frac{dV}{dr}=4\pi r^2$

$\frac{dr}{dt}=\frac{dV}{dt}\times\frac{dr}{dV}=3\times\frac{1}{4\pi r^2}$

$r=2, \quad \frac{dr}{dt}=3\times\frac{1}{4\pi 4}$

$=0.06\ \text{cm s}^{-1}$ to 2 d.pl.

193. (a) $f(3)=2(27)-5(9)-13(3)-6$

$=54-9-39-6=0$

$(x-3)$ is a factor of $f(x)$

(b) $f(-{}^1\!/_2)=-2({}^1\!/_8)-({}^1\!/_4)-{}^{13}\!/_2-6=0$

$(2x+1)$ is a factor of $f(x)$

(c) $(2x+1)(x-3)(x+2)=0$

$x=-2,-{}^1\!/_2$ or 3

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194. $$\frac{8-5x}{(1-x)^2(4-x)} \equiv \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(4-x)}$$

$$8-5x=A(1-x)(4-x)+B(4-x)+C(1-x)^2$$

$x=1 \quad 3=3B \Rightarrow B=1$

$x=4 \quad -12=9C \Rightarrow C=-\frac{4}{3}$

$x=0 \quad 8=4A+4-\frac{4}{3} \Rightarrow A=\frac{4}{3}$

$$f(x)=\frac{x}{(1-x)^2(4-x)}=\frac{4}{3(1-x)}+\frac{1}{(1-x)^2}-\frac{4}{3(4-x)}$$

$$f(x)=\tfrac{4}{3}(1-x)^{-1}+(1-x)^{-2}-\tfrac{4}{3}(4-x)^{-1}$$

$$f'(x)=\tfrac{4}{3}(-1)(-1)(1-x)^{-2}+(-2)(-1)(1-x)^{-3}$$

$$-\tfrac{4}{3}(-1)(-1)(4-x)^{-2}$$

$$=\frac{4}{3(1-x)^2}+\frac{1}{(1-x)^3}-\frac{4}{3(4-x)^2}$$

195.
$$\begin{array}{r} 2x^2+3x \\ 5x-1\ \overline{)10x^3+13x^2-3x+4} \\ -(10x^3-2x^2) \\ 15x^2-3x+4 \\ -(15x^2-3x) \\ 4 \end{array}$$

$2x^2+3x+\frac{4}{(5x-1)}$

196. $\frac{dy}{dx}=10^x \ln 10$

197. $\frac{2}{2}\ln|2x+1|+\frac{2}{-1\times 2}(2x+1)^{-1}$

$=\ln 4|2x+1|-\frac{1}{(2x+1)}$

198. $\frac{dy}{dx}=\frac{x^2}{4}-\frac{x}{2}+\frac{1}{2}$

$\frac{d^2y}{dx^2}=\frac{x}{2}-\frac{1}{2}$

Pt. of inflection $\frac{x}{2}-\frac{1}{2}=0$

$x=1,\ y=\frac{1}{3},\ \frac{dy}{dx}=\frac{1}{4}$ i.e. $\neq 0$

Just before $x=1$ say $x=0$ $\frac{d^2y}{dx^2}=-\frac{1}{2}$ i.e. –ve

Just after $x=1$ say $x=2$ $\frac{d^2y}{dx^2}=\frac{1}{2}$ i.e. +ve

$\frac{d^2y}{dx^2}$ changes sign $\frac{dy}{dx}\neq 0$

$\therefore (1, \frac{1}{3})$ is a non-stationary point of inflection.

199. $\overrightarrow{BA}=5\boldsymbol{i}+4\boldsymbol{j}+\boldsymbol{k}$, $\overrightarrow{BC}=\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$,
$\overrightarrow{AC}=-4\boldsymbol{i}-5\boldsymbol{j}+3\boldsymbol{k}$
$BA=\sqrt{42} \quad BC=\sqrt{18} \quad AC=\sqrt{50}$

cos(angle ABC) $=\frac{42+18-50}{2\sqrt{42}\sqrt{18}}=0.181848$

Angle ABC = 79.5°

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200. $x=\tan\theta \quad \frac{dx}{d\theta}=\sec^2\theta$

$$\int\frac{1}{1+x^2}dx=\int\frac{1}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$=\int\frac{\sec^2\theta}{\sec^2\theta}d\theta=\int 1 d\theta$$

$=\theta+c$

$=\arctan x+c$

201. $6x+6y\frac{dy}{dx}-5x\frac{dy}{dx}-5y=0$

$\frac{dy}{dx}=\frac{5y-6x}{6y-5x}$

$x=1, y=2 \quad \frac{dy}{dx}=\frac{4}{7}$

202. $\frac{dx}{dt}=-2\sin t \quad \frac{dy}{dt}=4\times 2\sin t\cos t$

$$\frac{dy}{dx}=8\sin t\cos t\times\frac{1}{-2\sin t}=-4\cos t$$

203. (a) $\cos 2x=2\cos^2x-1$, $\cos^2x=½(\cos 2x+1)$

$$\int\cos^2 x\, dx=\frac{1}{2}\int\cos 2x+1\, dx$$

$$=\frac{1}{2}(\frac{\sin 2x}{2}+x)+c=\frac{1}{4}\sin 2x+\frac{1}{2}x+c$$

(b) $u=10x \quad \frac{du}{dx}=10 \quad \frac{dv}{dx}=\sin x \quad v=-\cos x$

$$\int_0^{\pi}10x\sin x\, dx=-10x\cos x-\int-10\cos x\, dx$$

$$=\left[-10x\cos x+10\sin x\right]_0^{\pi}=(-10\pi(-1)+0)-(0)=10\pi$$

204. $\frac{dV}{dt}=-kV \quad k>0$

$$\int\frac{1}{V}dV=\int-k\, dt$$

$\ln V=-kt+c \quad t=0 \quad V=A \quad \therefore c=\ln A$

$\ln V=-kt+\ln A$

$\ln\frac{V}{A}=-kt \Rightarrow \frac{V}{A}=e^{-kt}$

Let $k=\lambda$

$\Rightarrow V=Ae^{-kt}$ i.e. $V=Ae^{-\lambda t}$

$t=8,\ \frac{3}{4}A=Ae^{-8\lambda} \Rightarrow \frac{3}{4}=e^{-8\lambda}$

$\Rightarrow \ln\frac{3}{4}=-8\lambda \Rightarrow \lambda=0.03596$

$V=\frac{1}{2}A \quad \frac{1}{2}A=Ae^{-0.03596\, t}$

$\ln\frac{1}{2}=-0.03596\, t \Rightarrow t=19$ sec

205. Assume you can find a value of $x>0$ where

$x+\frac{1}{x}<2$

multiply everything by x $(x>0)$

gives $x^2+1<2x$

produce quadratic inequality $x^2-2x+1<0$

factorise $(x-1)^2<0$

This is impossible for any value of x.

$\therefore$ the assumption was false

$\therefore$ the original statement is true.

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206. $\frac{3-x}{2+x}\equiv\frac{A}{1}+\frac{B}{2+x}$

$3-x=A(2+x)+B$

$x=-2 \quad 5=B$

$x=0 \quad 3=2A+5 \quad A=-1$

$\frac{3-x}{2+x}\equiv-1+\frac{5}{2+x}$

207. (a) $\int_1^4\sqrt{x}\, dx=\left[\frac{2x^{\frac{3}{2}}}{3}\right]_1^4=\frac{2\times 8}{3}-\frac{2}{3}=\frac{14}{3}$

(b) $x=1 \quad y=\sqrt{1}=1$
$x=2.5 \quad y=\sqrt{2.5}=1.58113883$
$x=4 \quad y=\sqrt{4}=2$ $\quad$ h = 1.5
Area ≈ 1.5/2 [1 + 2×1.588113883 + 2] ≈ 4.62

(c) percentage error $=\frac{\frac{14}{3}-4.621708245}{\frac{14}{3}}\times 100$

$=0.963\%$

208. (a) $\frac{dx}{dt}=2t \quad \frac{dy}{dt}=3t^2 \quad \frac{dy}{dx}=\frac{3t^2}{2t}=\frac{3t}{2}$

At t = 2 gradient = 3, $x=4, y=8$

$y-8=3(x-4)$

$y=3x-4$

(b) Area under curve $=\int_0^2 t^3 2t\, dt$

$=\int_0^2 2t^4\, dt=\left[\frac{2t^5}{5}\right]_0^2=\frac{64}{5}$

Area of triangle $=\frac{1}{2}\times\frac{8}{3}\times 8=\frac{32}{3}$

Required area $=\frac{64}{5}-\frac{32}{3}=\frac{32}{15}=2\frac{2}{15}$ sq units

(c) $3t^3=2t^2+1$ $\quad$ t = 1 fits by trial and error.
Point of intersection (1, 1)

209. $f(x)=x^3+x-3 \quad f(1)=-1 \quad f(2)=7$
There is a change of sign and therefore a root between $x=1$ and $x=2$
$x_1=1 \quad x_2=1.25992105$
$x_3=1.202789564 \quad x_4=1.215811678$
$x_5=1.212868073 \quad x_6=1.213534715$
$\therefore x=1.21$ to 3 sig. fig.

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210. $$\frac{4-x}{(2+x)(1-x)}=\frac{A}{(2+x)}+\frac{B}{(1-x)}$$

$$4-x=A(1-x)+B(2+x)$$

$x=1 \quad 3=3B \quad B=1$

$x=-2 \quad 6=3A \quad A=2$

$$\frac{4-x}{(2+x)(1-x)}=\frac{2}{(2+x)}+\frac{1}{(1-x)}$$

$$=2(2+x)^{-1}+(1-x)^{-1}$$

$2(2+x)^{-1}=2\times 2^{-1}(1+\frac{x}{2})^{-1}$

$=1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2.1}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3.2.1}\left(\frac{x}{2}\right)^3+..$

$=1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+..$

$(1-x)^{-1}=(1+(-x))^{-1}$

$=1+(-1)(-x)+\frac{(-1)(-2)}{2.1}(-x)^2+\frac{(-1)(-2)(-3)}{3.2.1}(-x)^3+..$

$=1+x+x^2+x^3+..$

$\frac{4-x}{(2+x)(1-x)}=1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+..+1+x+x^2+x^3+..$

$=2+\frac{1}{2}x+\frac{5}{4}x^2+\frac{7}{8}x^3+..$

Valid for $-1<x<1$

211. $\int\frac{1}{y}dy=\int\sin 2x\, dx$

$\ln y=-\frac{\cos 2x}{2}+c$

$y=e^{-\frac{\cos 2x}{2}+c}=e^{-\frac{\cos 2x}{2}}e^c=Ae^{-\frac{\cos 2x}{2}}$

$x=\frac{\pi}{4} \quad y=5 \quad 5=A$

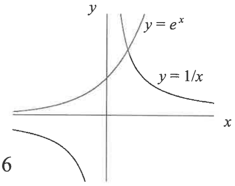
$y=5e^{-\frac{\cos 2x}{2}}$

212. (i) $\overrightarrow{AB}=\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$
(ii) $AB=\sqrt{11} \quad OA=\sqrt{11} \quad OB=\sqrt{24}$

cos(angle OAB) $=\frac{11+11-24}{2\sqrt{11}\sqrt{11}}=-0.0909$

Angle OAB = 95.2°

213.



$f(x)=e^x-1/x$
$f(0.5)=-0.351$
$f(0.6)=0.155$
Change of sign.
$\therefore$ Root between
$x=0.5$ and $x=0.6$

$x_1=0.6,\ x_2=0.566205...,\ x_3=0.567142....,$
$x_4=0.567143$
f(0.5665) = – 0.003.. f(0.5675) = 0.0017..
Change of sign. $\therefore x=0.567$ to 3 dec. pl.