Redhill A-Level Maths

Workbook A2 Pure 1

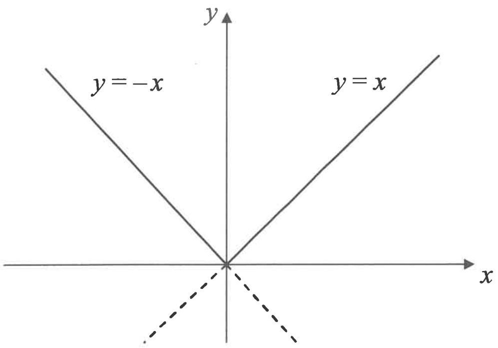
ALGEBRA and FUNCTIONS

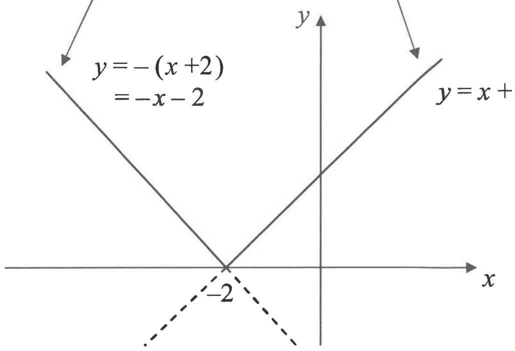
The modulus of a linear function:

$y = |f(x)|$ Draw the graph of $y = f(x)$ and reflect in the x axis (which is $y = -f(x)$). The graph of $y = |f(x)|$ is the part of both graphs above and touching the x axis.

Notice that although this is the graph of $y = |x + 2|$, this straight line section is part of the graph $y = -x - 2$

And this straight line section is part of the graph $y = x + 2$





$y = |x|$ (Not including the dotted lines.)

$y = |x + 2|$ (Not including the dotted lines.)

1. Sketch the graphs $y = |x - 1|$ and $y = 6$
   Find the solutions of the equation $|x - 1| = 6$

Sketch $y = x - 1$ and reflect in the x axis.

Sketch $y = 6$

Label the points where the graphs touches/cuts the axes.

Solve for one point (where $x - 1 = 6$) and hence find one value of x.

Solve for the second point. (Where $-(x - 1) = 6$)

Write down the two solutions.

Note: The intersection points give the upper and lower bounds for $|x - 1| \leq 6$ These are $1 - 6 \leq x \leq 1 + 6$
Generalising $|x - a| \leq b$ gives $a - b \leq x \leq a + b$ for the upper and lower bounds.

2. Sketch the graphs $y = |2x - 1|$ and $y = x + 2$
   Find the solutions of the equation $|2x - 1| = x + 2$

Sketch $y = 2x - 1$ and reflect in the x axis. $y = |2x - 1|$ is that part of both graphs above and touching the x axis.

Sketch $y = x + 2$

Label the points where the graphs touches/cuts the axes.

The solutions of $|2x - 1| = x + 2$ are the points of intersection of the two graphs.
Solve for one point. This is where $2x - 1 = x + 2$ and hence obtain one value of x.

Solve for the second point. Where $-(2x - 1) = x + 2$

Write down the two solutions.

3. Find the solutions of the equation $|2x+3| < -x$

Sketch $y = 2x + 3$ and reflect in the x axis. $y = |2x + 3|$ is that part of both graphs above and touching the x axis.

Sketch $y = -x$

Label the points where the graphs touches/cuts the axes.

The solutions of $|2x+3| > -x$ are the points of intersection of the two graphs. Solve for one point. This is where $2x + 3 = -x$ Hence obtain one value of x.

Solve for the second point. Where $-(2x + 3) = -x$

Write down the solutions.

4. Solve $|x-5| > 3$

Always sketch the graphs so that you know you are using the correct equations.

5. Solve $|x-1| \leq |2x+3|$

Always sketch the graphs so that you know you are using the correct equations.

Functions - Definitions and notation:

A function is a relationship that maps <u>one</u> value from a set of numbers to <u>one</u> value in a second set of numbers.

The set of input numbers is called the ***domain***.

The set of output numbers is called the ***range (or co-domain)***.

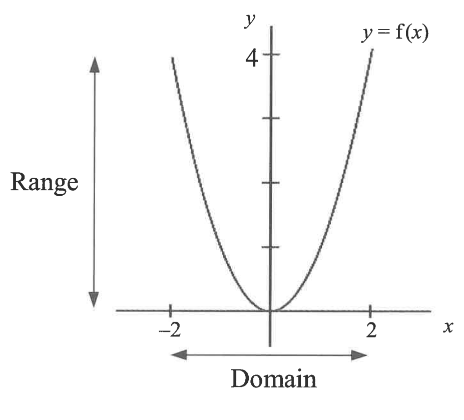
e.g. the function x^2 can be written $y = x^2$ or $f(x) = x^2$ or $f: x \to x^2$

The set of input numbers can be restricted (or not - then the domain will be for all x).

$f(x) = x^2$, for $-2 \leq x \leq 2, x \in \mathbb{R}$

This is part of the definition of a function. It should always be given so that you know for what values of x the function is valid (the domain). If the domain is not given, then assume it is for all values real values of x. $x \in \mathbb{R}$ means that x is a member of the set of real numbers.

Sketch the graph $y = f(x)$ if possible.



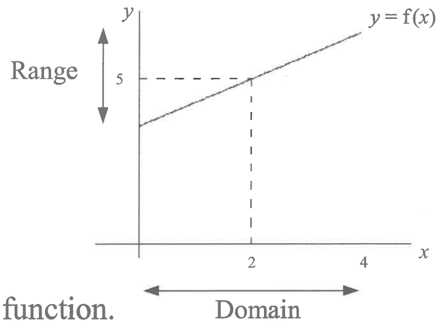
It is then easy to 'see' the domain and range.
i.e. the possible x values and the possible y values for the graph.

One-to-one functions map one value from the domain to one value in the range.

e.g. $f(x) = x + 3$, $0 \le x \le 4, x \in \mathbb{R}$

$f(2) = 5$ i.e. 2 maps to 5

The inverse mapping will map 5 to 2. This will be a one-to-one mapping and therefore a function.



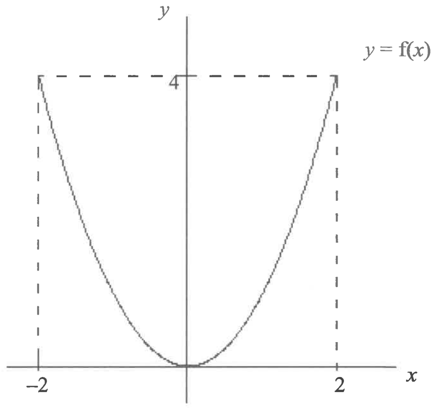
Each one-to-one function has an inverse function.

Many-to-one functions map one value from the domain to one value in the range, but two or more values in the domain will map to the same value in the range.

e.g. $f(x) = x^2$, $-2 \le x \le 2, x \in \mathbb{R}$
Here – 2 and 2 both map to 4.

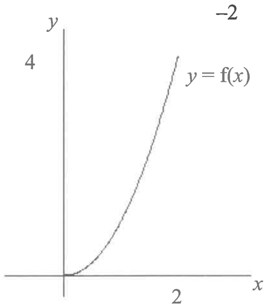
The inverse mapping will map 4 to both – 2 and 2. i.e. **one-to-many**. This is **not a function** (see definition above).

Many-to-one functions do not have inverse functions, unless the domain is restricted so that the function is one-to-one.



e.g. $f(x) = x^2$, $0 \le x \le 2$, $x \in \mathbb{R}$

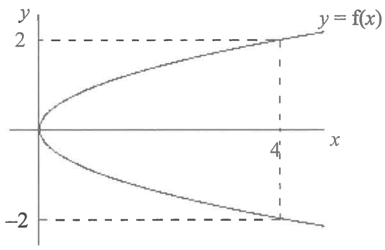
This is now a one-to-one function and so has an inverse function.



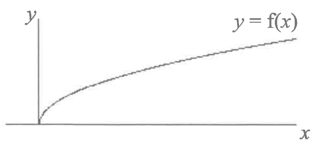
One-to-many mappings are not functions. Here one value in the input set maps to more than one value in the output set.
If the mapping is restricted, so that we have a one-to-one mapping, then this will be a function.

e.g. $f(x) = \pm\sqrt{x}$,
$x \ge 0, x \in \mathbb{R}$

Here 4 maps to 2 and –2



Restricting the mapping to $f(x) = +\sqrt{x}$, $x \ge 0, x \in \mathbb{R}$ means we have a one-to-one mapping and therefore a function.



Composite functions:
Consider the function to change Fahrenheit to Celsius:

$$C = \frac{5}{9}(F - 32) \quad \text{or in function notation} \quad f(t) = \frac{5}{9}(t - 32) \quad \text{or} \quad f : t \to \frac{5}{9}(t - 32)$$ where t is the temperature.

And to change Celsius to Kelvin (another way of measuring temperature):

$$K = C + 273 \quad \text{or in function notation} \quad g(t) = t + 273 \quad \text{or} \quad g : t \to t + 273$$

°F	f →	°C	g →	°K
212	→	100	→	373
113	→	45	→	318
68	→	20	→	293
32	→	0	→	273

So this means do f first, then g.

So $f(113) = 45$ and $g(45) = 318$ OR $g(f(113)) = 318$ which is written $gf(113) = 318$

For $f(x) = 2x + 3$ and $g(x) = x - 2$ the composite function (g first then f) is written $f(g(x)) = fg(x)$
Here the input values are first substituted into the $g(x)$ function and the result substituted into the $f(x)$ function.
To evaluate the composite function for a particular value of x you write:
$fg(3) = f(3 - 2) = f(1) = 2 \times 1 + 3 = 5$ (when $x = 3$) Here you must remember to calculate the g function first.
To obtain the function $fg(x)$ you replace each x in f with $x - 2$ i.e. $fg(x) = f(x - 2) = 2(x - 2) + 3 = 2x - 1$

There are also functions that are not valid for every x, but that can be valid if you restrict the values of x. For example, take $f(x) = \sqrt{x-1}$, $g(x) = 2x - 5$. Now only positive numbers, or zero, have real square roots. So f is defined only for numbers $x - 1 \geq 0$ i.e. $x \geq 1$. Therefore $fg(x)$ can have a value only if $g(x)$ is greater than or equal to 1. You can work out that $g(x) \geq 1$ only when $2x - 5 \geq 1$ i.e. $x \geq 3$. So the composed function $fg(x)$ can be defined only for $x \geq 3$ and therefore the domain of the function fg is $x \geq 3, x \in \mathbb{R}$

6. If $f(x) = x^2$ and $g(x) = 2x - 4$ find:

(i) $fg(5) = f(\quad) =$

(i) g first. Put $x = 5$ into $2x - 4$. Then this answer into x^2

(ii) $fg(-2) = f(\quad) =$

(iii) $fg(x) = f(\quad) =$

(iii) Putting x into $2x - 4$ gives $2x - 4$ Then replace the x in the f function with $2x - 4$.

(iv) $gf(5) = g(\quad) =$

(v) $gf(-2) = g(\quad) =$

(vi) $gf(x) = g(\quad) =$

7. If $f(x) = \ln x$, $g(x) = x^2 - 1$ and $h(x) = 3x + 2$ find:

(i) $h(2) =$ (ii) $gh(2) =$

(iii) $fgh(2) =$ (iv) $fgh(-2) =$

(v) $gh(x) =$

(vi) $fgh(x) =$

8. If $f(x) = 4x - 3$, $g(x) = 5 - x$ find:

(i) $f(3) =$ (ii) $g(3) =$ (iii) $fg(3) =$ (iv) $gf(3) =$

(v) $fg(x) =$ (iv) $gf(x) =$

9. If $f(x) = 4x - 3$, $g(x) = \ln x$ find the possible values of x for which the function $gf(x)$ is valid.

10. If $f(x) = 3 - 2x$, $0 \leq x \leq 5, x \in \mathbb{R}$, $g(x) = \ln(2 - x)$, $-7 < x < 2, x \in \mathbb{R}$ find the possible values of x for which the function $gf(x)$ is valid.

Inverse functions:

If a function f maps x onto y then the *inverse function* f^{-1} maps y onto x

For $f(x) = 2x + 3$

x	f	y	f^{-1}	x
1	→	5	→	1
2	→	7	→	2
3	→	9	→	3
4	→	11	→	4

etc.

This also shows that if you put an x value into the function and then put that result into the inverse function you get back to x.
So $ff^{-1}(x) = x$
Similarly $f^{-1}f(x) = x$

The function $f : x \to 2x + 3$ (i.e. times by 2 and then add three) will map 1 onto 5 and so on.
The inverse function is one that will map 5 onto 1 and so on.
The inverse function will be 'subtract 3 and then divide by 2' (the inverse operations).
i.e. $f^{-1} : x \to \frac{x-3}{2}$ or $f^{-1}(x) = \frac{x-3}{2}$

To obtain inverse functions, start with your function equal to y.

$$y = 2x + 3$$

swap over the xs and ys

This is because the range (the output values) is to become the domain (the input values) for the inverse function. (You can do this later after the equation has been rearranged to give x =....)

$$x = 2y + 3$$

Now rearrange to give y in terms of x.

$$x - 3 = 2y$$

$$\frac{x-3}{2} = y$$

This is the inverse function. i.e. $f^{-1}(x) = \frac{x-3}{2}$

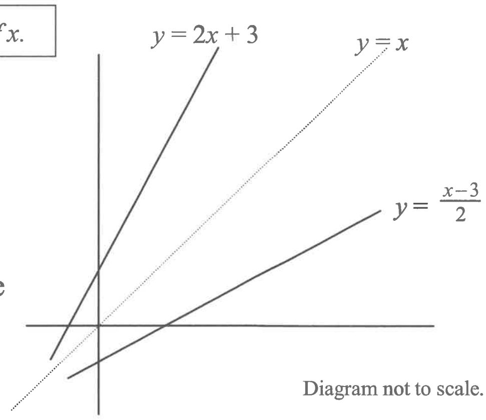


Diagram not to scale.

If you plot $y = f(x)$ and $y = f^{-1}(x)$ on the same grid (with the same scale on both axes) you can see that the graph of $f^{-1}(x)$ is a reflection of the graph $f(x)$ in the line $y = x$ (and vice versa).

11. If $f(x) = \frac{3}{4}(x-2)$ find $f^{-1}(x)$	
Complete: $y = \frac{3}{4}(x-2)$	*Write y =*
$x = \frac{3}{4}(y-2)$	*Swap xs and ys*
$=$	*Multiply both sides by 4 and divide both sides by 3.*
$y =$	*Adding 2 gives y in terms of x.*
$f^{-1}(x) =$	*Write as the inverse function.*
12. If $f(x) = \frac{1}{x+3}$ find $f^{-1}(x)$	*Write y =* *Swap xs and ys.* *Multiply both sides by (y + 3).* *Divide both sides by x (or multiply out the bracket).* *Subtract 3 from both sides (or subtract 3x and divide by x).* *Write as the inverse function.*

13. Complete: If $f(x) = \frac{x+1}{x-1}$ find $f^{-1}(x)$

Write y =

Swap xs and ys

Multiply both sides by y – 1

Multiply out the bracket.

Collect all terms with y in on one side and all other terms on the other side of the equation.

Take out y as a factor.

Divide both sides by the other factor.

Write as the inverse function. (This function is its own inverse!)

Reminder-transformations of graphs:

A curve with equation $y = f(x + k)$ is a translation of $y = f(x)$ by vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

A curve with equation $y = f(x) + k$ is a translation of $y = f(x)$ by vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$

A curve with equation $y = af(x)$ is a stretch of factor a parallel to the y axis (away from the x axis).

A curve with equation $y = f(ax)$ is a stretch of factor $\frac{1}{a}$ parallel to the x axis (away from the y axis).

A curve with equation $y = f(-x)$ is a reflection of $y = f(x)$ in the y axis.
A curve with equation $y = -f(x)$ is a reflection of $y = f(x)$ in the x axis.

Combining transformations:

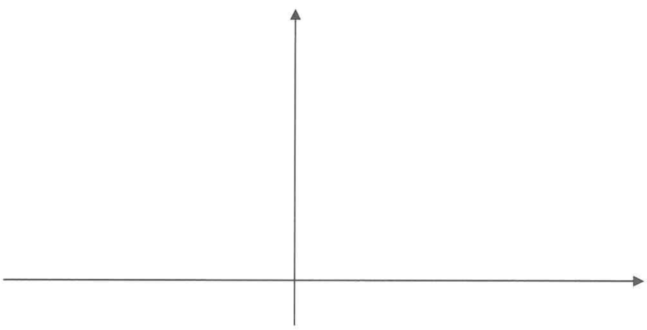
When combining functions the order that the transformations are applied and how they are applied can be important. Consider $f(x) = x^2$ and the graph of $y = f(2x - 4)$ You replace the x with $2x - 4$ to draw $y = (2x - 4)^2$ If you stretch $y = x^2$ by factor 2 parallel to y axis away from the x axis and a translation of 4 to the right is not the same graph - so it is wrong! Check this yourself.

In general:

1. Write in the form $y = af(b(x + c)) + d$
2. Stretches and reflections should be applied first, in any order using a and/or b.
3. Translations should be applied last, in any order using c and/or d.

So $4f(2x + 6) + 5 = 4f(2(x + 3)) + 5$ i.e. A horizontal stretch factor ½ and a vertical stretch factor 4, followed by a horizontal translation of 3 to the left and a vertical translation 5 up.

14. Sketch $y = 2e^{-x}$



Here you can do this in 3 stages.

In pencil sketch $y = e^x$ $[y = f(x)]$

(Label where the graph cuts the y axis.)

Then - in pencil sketch $y = 2e^x$ (Stretch factor 2 parallel to the y axis away from the x axis.) $[y = 2f(x)]$
(Label where the graph cuts the y axis.)

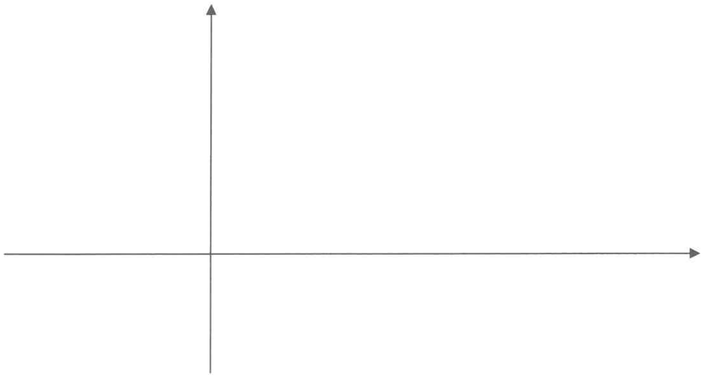
In pen sketch $y = 2e^{-x}$ (Reflection in the y axis.) $[y = 2f(-x)]$

Check your result on a graphical calculator.

15. If a point $(6, 12)$ is on the graph $y = f(x)$, find the coordinates of the new point after the following transformations:

(i) $y = 2f(3x) + 1$ (ii) $y = 2f(3x + 1) - 5$ (iii) $y = \frac{1}{4}f(\frac{1}{2}x - 1) + 3$

16. Sketch $y = 3\cos 2x + 4$ for $-90^\circ \le x \le 180^\circ$

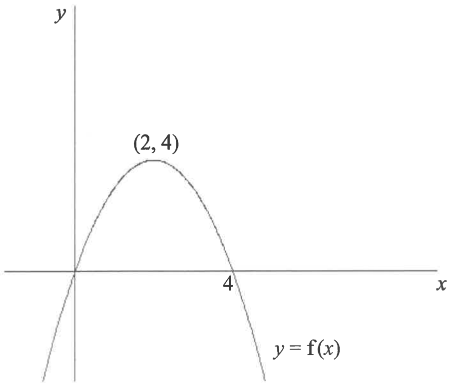


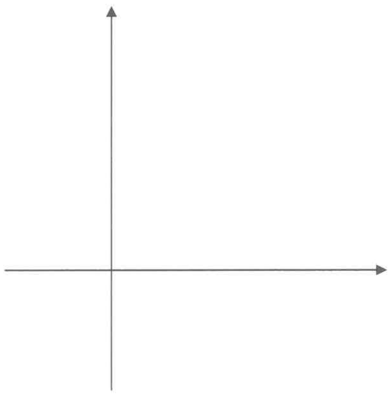
Here you can do this in 4 stages.

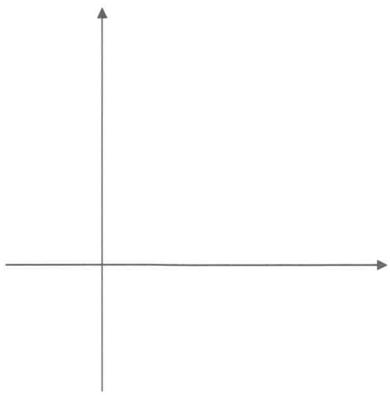
	If f(x) = cos x then you have sketched:
Sketch y = cos x	$y = f(x)$
Sketch y = cos 2x	$y = f(2x)$
Sketch y = 3 cos 2x	$y = 3f(2x)$
Sketch y = 3 cos 2x + 4	$y = 3f(2x)+4$

Use a pencil - When you sketch y = cos x you will need to allow for the stretches and the translation.

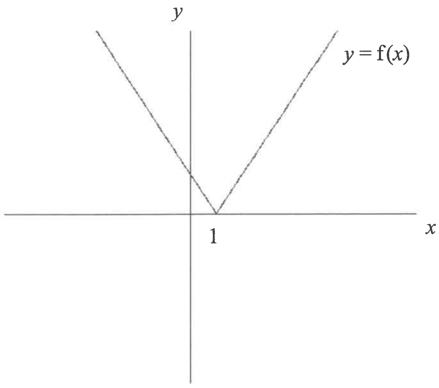
17. Here is a graph of $y = f(x)$. On separate diagrams sketch $y = 2f(2x)$ and $y = 2f(\frac{1}{2}x)$. Label the graphs.

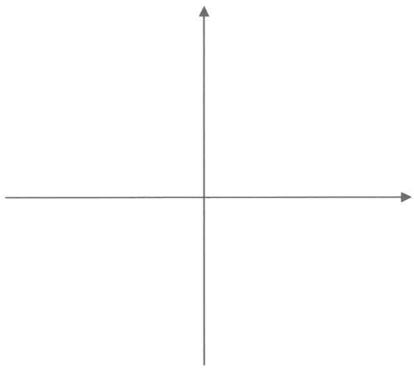


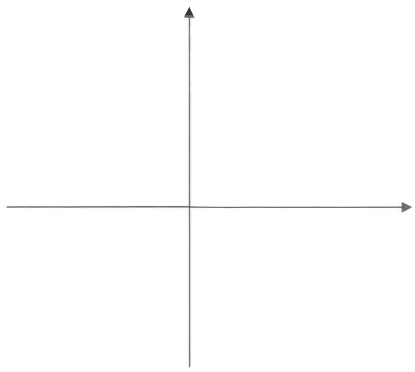




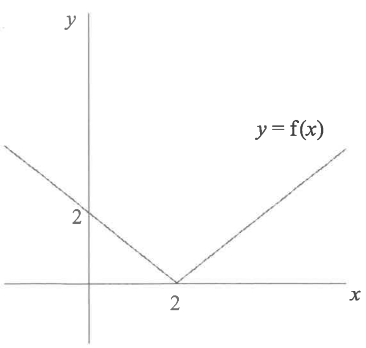
18. Here is a graph of $y = f(x)$ where $f(x) = |x - 1|$. On separate diagrams sketch $y = -f(x + 3)$ and $y = -f(-x)$ Label the graphs and label where they cut/touch any axes.

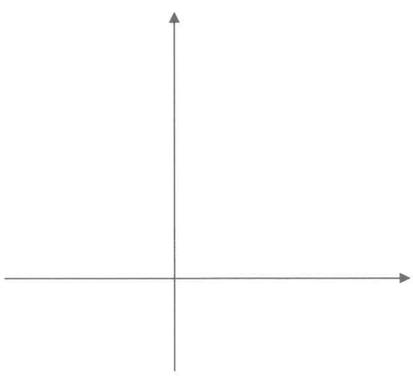


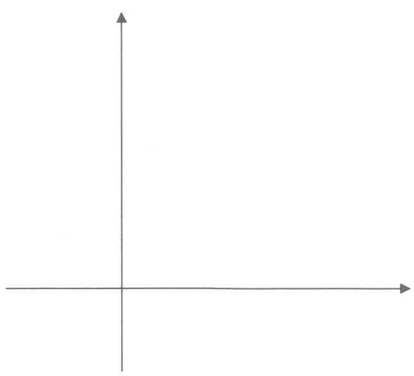




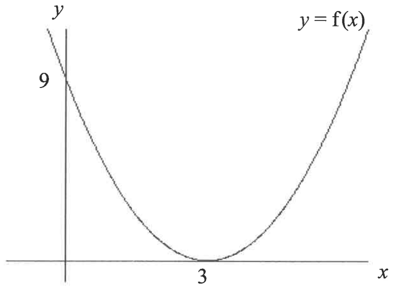
19. Here is a graph of $y = f(x)$ where $f(x) = |x - 2|$. On separate diagrams sketch $y = 3f(x) + 1$ and $y = f(\frac{1}{2}x) + 1$. Label the graphs and label where they cut any axes and the minimum point.

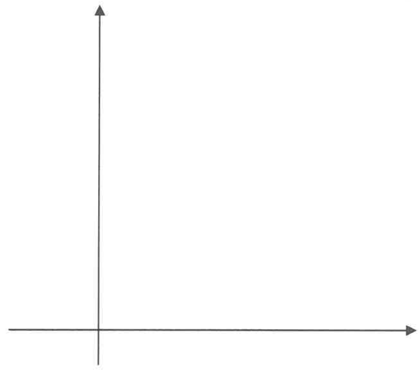
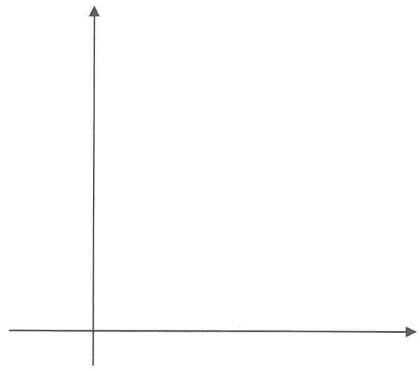




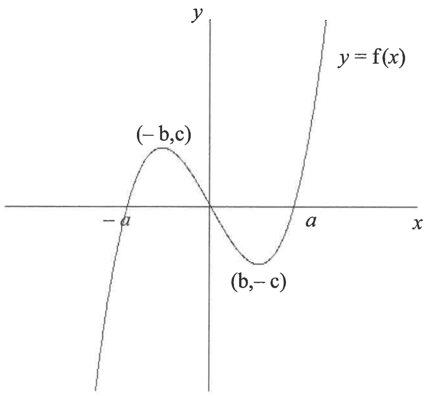


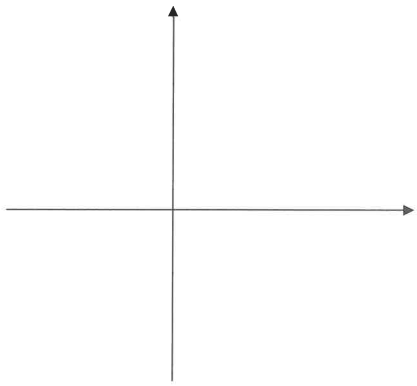
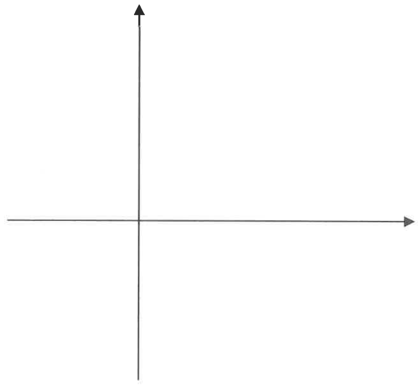
20. Here is a graph of $y = f(x)$. On separate diagrams sketch (i) $y = f(3x)$ (ii) $y = 3f(3x)$
Label the graphs and label where they cut/touch any axes.



21. Here is a graph of $y = f(x)$. On separate diagrams sketch $y = 2f(2x)$ and $y = 2f(\frac{1}{2}x)$. Label the graphs and label the important points in terms of a, b and c.



22. Sketch $y = \ln(2x + 1)$

Here think of the function for $f(x) = \ln x$
To sketch the transformed graph write in the form $y = af(b(x + c)) + d$

You are replacing the x in f with 2x+1

So $y = f(2x + 1)$ and rewrite in the form $y = af(b(x + c)) + d$

Hence you can apply the transformations on $y = f(x) = \ln x$

Note: To transform $y = \cos x$ to $y = \cos(2x + \pi)$ you would use $y = \cos(2(x + \pi/2))$ and so on for other functions.

Note: Q 3 and 6 - Your sketches may look similar to the original but as long as you label the important points (max and min and where it crosses the axes) it will be fine as they are scaled diagrams.

Use of functions in modelling:

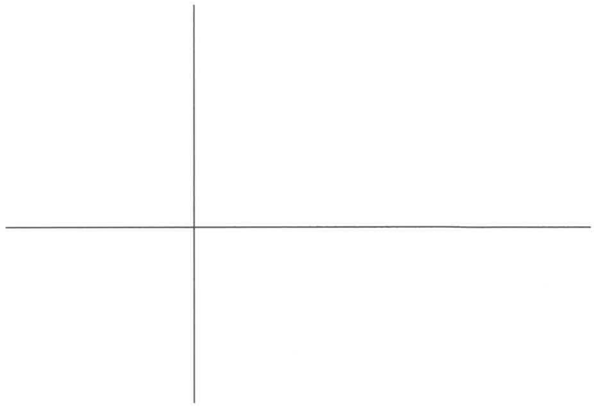
23. It takes approximately x hours to fly from London to Cancun, Mexico, at y mph. Write down an equation to model this flight. If it takes 10 hours at 500 mph how long would it take at 600 mph? State one criticism of this model.

24. The number of days to complete a task is inversely proportional to the number of people working on the task. If it takes 4 people 120 days to complete a task, how many people do you need to complete the task in 30 days? State one criticism of this model.

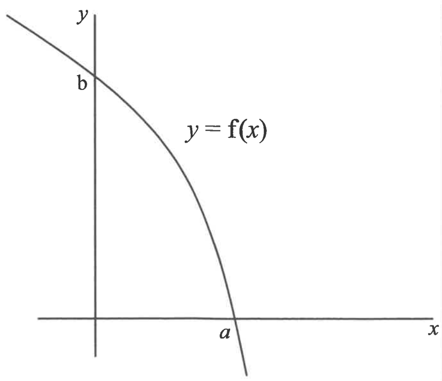
Also see book 1 of 6 page 30 for inverse proportion modelling.
See page 35 for Trig functions in modelling.
See book 5 of 6 for modelling with growth and decay functions and parametric equations.

Mixed questions 1

25. If $f(x) = 2x - 1$ find $f^{-1}(x)$ and sketch $y = f(x)$ and $y = f^{-1}(x)$ on the grid to verify that $f(x)$ and $f^{-1}(x)$ are reflections of each other in the line $y = x$. Label where each of the graphs cuts the axes.



26. Given $y = f(x)$ as shown, sketch the graphs of the following functions, labelling the intersections with the axes in each case.
(i) $y = 2f(3x)$ (ii) $y = f(-\frac{1}{2}x)$ (iii) $y = -f(2x)$ (iv) $y = f^{-1}(x)$



27. If $f(x) = x^2$, $g(x) = 5x - 2$ and $h(x) = 2x + 3$ find (i) f(3) (ii) fg(2) (iii) fgh(1) (iv) fg(x) (v) gh(x)

Remember: To obtain the function f(5x – 2) you replace each x in f with 5x – 2

28. The graph of $y = e^x$ is transformed into the graph $y = \ln x$ Describe this transformation geometrically.

29. The graph of f(x) is plotted. f(x) is a one-to-one function.

(a) Describe the transformations which would produce

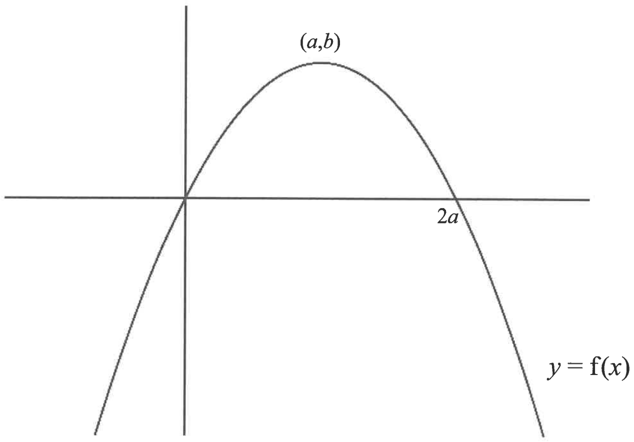
(i) $y = -f(x)$

(ii) $y = f^{-1}(x)$

(iii) $y = f(-x)$

(b) If the graph of f($-x$) is the same as the graph of f(x) what information do you know about f(x)?

30. The graph of $y = f(x)$ is shown here.
On the same grid sketch $y = f(2x + 2a) - b$



31. $f(x) = 3x - 1$ $g(x) = x^2$
Find fg(x), gf(x) and ff(x) in terms of x

32. $f(x) = \dfrac{2x}{x^2+1}$ $g(x) = \dfrac{1}{x}$ Show that $fg(x) = f(x)$

SEQUENCES AND SERIES

Sequences:

A sequence is a set of numbers. E.g. 3, 6, 11, 18, 27, 38,…
Sometimes you can 'see' or 'work out' the n^{th} term. Here it is $n^2 + 2$

You may be given a sequence $u_1, u_2, u_3, u_4, \ldots$ defined:

$u_n = 5n + 2$ $(n = 1, 2, 3, \ldots)$
So the 1^{st} term is $u_1 = 5\times1 + 2 = 7$ $(n = 1)$
2^{nd} term $u_2 = 5\times2 + 2 = 12$ $(n = 2)$
$u_3 = 5\times3 + 2 = 17$ $(n = 3)$
etc

This type of formula gives the n^{th} term and the values of n for which it is valid.

OR

$u_1 = 5$, $u_{n+1} = u_n + 3$ $(n = 1, 2, 3, \ldots)$
So the 1^{st} term is $u_1 = 5$
2^{nd} term $u_2 = u_1 + 3 = 5 + 3 = 8$ $(n = 1)$
3^{rd} term $u_3 = u_2 + 3 = 8 + 3 = 11$ $(n = 2)$
$u_4 = u_3 + 3 = 11 + 3 = 14$ $(n = 3)$
etc.

This form for generating a sequence requires the first term, a rule and the values of n for which it is valid.

Given that a_n is the nth term of a sequence, then if $a_{n+1} > a_n$, for all n, the sequence is said to be **increasing**.

Given that a_n is the nth term of a sequence, then if $a_{n+1} < a_n$, for all n, the sequence is said to be **decreasing**.

Some sequences are neither increasing or decreasing.
A **periodic sequence** has terms that form a repeating pattern. E.g. 1,2,5,1,2,5,1,2,5….
The number of repeated terms is called the **period** or **order**.

Series:

A series is the sum of the terms of a sequence. E.g. 6 + 11 + 18 + 27 + ……

The summation notation: $\sum_{i=1}^{i=5} 2i+1$

If a series is given in this form it is often useful to write out the first few terms of the series. It can sometimes show you what type of series you have.

$$\sum_{i=1}^{i=5} 2i+1 = (2\times1+1)+(2\times2+1)+(2\times3+1)+(2\times4+1)+(2\times5+1)$$
$$= 3 + 5 + 7 + 9 + 11$$

A convergent series is one that approaches a particular value. E.g. 1 + 0.5 + 0.25 + 0.125 + 0.0625 +…..
the sum approaches 2

A divergent series does not converge to a value. E.g. 2 + 4 + 8 + 16 + ……
the sum increases as each term is added.

33. State whether the following sequences are increasing, decreasing or periodic. (a) $u_n = 2^n$ (b) $a_n = (-1)^n$ (c) $u_n = 1/(n+1)$ (d) $x_n = 1 - 2n$	
34. Write down the first five terms and the 20[th] term of each of the following sequences: (a) $x_n = 5n + 2 \quad (n = 1, 2, 3, ...)$ (b) $u_n = 2^{n-1} \quad (n = 1, 2, 3, ...)$ (c) $a_n = (-1)^n n^2 \quad (n = 1, 2, 3, ...)$	*First term n = 1* *So $x_1 = 5\times1 + 2 = ...$* *Then n = 2 etc.*
35. Write down the first five terms of each of the following sequences: (a) $x_1 = 100, \quad x_{n+1} = x_n - 15 \quad (n = 1, 2, 3, ...)$ (b) $u_1 = 1, \quad u_{n+1} = \frac{1}{2}u_n + 1 \quad (n = 1, 2, 3, ...)$ (c) $v_0 = 10, \quad v_{n+1} = 2v_n + 5 \quad (n = 0, 1, 2, 3, ...)$	*n = 1* *so $x_{1+1} = x_1 - 15$* *i.e $x_2 =$ ....etc.* *(c) Careful - start with n = 0*
36. Write down each of the following as a series and find the sum: (a) $\sum_{i=1}^{5} 3i - 1$ (b) $\sum_{n=1}^{4} 2^n + 1$ (c) $\sum_{i=3}^{8} i^2$	*i = 1 3×1–1* *then i =2 and so on. Add the terms together.* *(c) Careful - start with i = 3*

Arithmetic series:

Sometimes called an Arithmetic Progression (or A.P. for short).
Arithmetic series are of the form

$a + (a+d) + (a+2d) + (a+3d) + \ldots\ldots$ where a is the first term and d is the common difference.

$\downarrow$ 1st, $\downarrow$ 2nd, $\downarrow$ 3rd, $\downarrow$ 4th terms

The n[th] term: $n^{th} \text{ term} = a + (n-1)d$

The sum to n terms, $S_n = \frac{1}{2}n(a + l)$ OR $S_n = \frac{1}{2}n[2a + (n-1)d]$

l is the last term e.g. If there are ten terms the last term can be found from the n[th] term formula above with $n = 10$

Learn how to derive this formula from your set text or notes.

Example:
Find the sum to twelve terms of the series $1 + 3 + 5 + 7 + 9 + \ldots\ldots\ldots$

$a = 1$, $d = 2$ (difference of 2 between consecutive numbers)

Using $S_n = \frac{1}{2}n(a + l)$ **OR** Using $S_n = \frac{1}{2}n[2a + (n-1)d]$

12^{th} term $= 1 + 11 \times 2 = 23$

$S_{12} = \frac{1}{2} \times 12 \times (1 + 23)$
$= 6 \times 24$
$= 144$

$S_{12} = \frac{1}{2} \times 12 \times [2 \times 1 + (12 - 1)2]$
$= 6[2 + 22]$
$= 144$

37. Find the 15 th term in the sequence 2, 9, 16, 23,…	*15^{th} term $a + 14d$*
38. Find the sum of the first 25 odd numbers.	*Write down a and d and use formula for S_n*
39. If 50 is the n th term in the sequence 2, 8, 14,…..Find n.	*n^{th} term $a + (n - 1)d$*
40. 4, p , q, 31 are the first four terms in an arithmetic sequence. Find p and q.	*Use expression for 4 th term = 31 to find d.*
41. The 3 rd term of an arithmetic sequence is 17 and the 7 th term is 5 Find the first term and the common difference.	*Use formula for k th term.* *Simultaneous equations.*
42. The sum to n terms of a series is $n(n + 4)$. Find the first three terms.	*Find the sum to 1 term.* *Find the sum to 2 terms (and hence calculate the 2 nd term by subtraction).* *Find the sum to 3 terms (and hence calculate the 3 rd term by subtraction).*

Note: The sum of the first n Natural numbers i.e. $1 + 2 + 3 + 4 + 5 + \ldots\ldots\ldots\ldots\ldots + n$
(an A P with a = 1 and d = 1) is given by

$$S_n = \tfrac{n}{2}(n+1)$$

So the sum of the first 20 Natural numbers is $S_{20} = \frac{20}{2}(20+1) = 10 \times 21 = 210$

43. Find the sum of the first 30 Natural number.	
44. Find the sum of the integers from 100 to 200 inclusive.	*Careful! Sum to 200 take away sum to 99*
45. Evaluate $\sum_{i=1}^{25} i$	*Write out the first few terms. You will then see that it is just the sum of Natural numbers.*

46. Evaluate $\sum_{i=40}^{60} i$	*Write out the first few terms. Careful! Start at 40.*

Geometric series:

Sometimes called a Geometric Progression (or G.P. for short).
This is a series of the form

$$\underset{1^{st}}{\underset{\downarrow}{a}} + \underset{2^{nd}}{\underset{\downarrow}{ar}} + \underset{3^{rd}}{\underset{\downarrow}{ar^2}} + \underset{4^{th}}{\underset{\downarrow}{ar^3}} + \ldots\ldots \text{ terms}$$

where a is the first term and r is the common ratio (think of common multiplier).

The n^{th} term is given by $\boxed{n^{th} \text{ term} = a\,r^{\,n-1}}$

The sum to n terms $\boxed{S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1}}$

Learn how to derive this formula from your set text or notes.

Sum to infinity $\boxed{S_\infty = \dfrac{a}{1-r} \quad \text{for } -1 < r < 1}$

Note: Will only converge to a sum if $-1 < r < 1$ or $|r| < 1$

Example 1

$1 + 2 + 4 + 8 + 16 + \ldots\ldots\ldots$

$a = 1$, $r = 2$ (multiplier of 2 between consecutive numbers)

12^{th} term $= 1 \times 2^{11} = 2048$

The sum to seven terms

$$S_7 = \frac{1(1-2^7)}{1-2} = \frac{-127}{-1} = 127$$

OR

$$S_7 = \frac{1(2^7-1)}{2-1} = \frac{127}{1} = 127$$

Example 2

The nth term of a geometric series is 6804.498 If the first term is 8 and the common ratio is 1.2 find n.

$$6804.498 = 8 \times 1.2^{n-1}$$
$$850.56225 = 1.2^{n-1}$$

The unknown is a power - so take logs.

$$\log 850.56225 = \log 1.2^{n-1}$$
$$\log 850.56225 = (n-1)\log 1.2$$
$$\log 850.56225 \div \log 1.2 = n - 1$$
$$n - 1 = 37$$
$$n = 38$$

47. Work out the 15^{th} term of the geometric sequence 1, 5, 25, 125, …..	*Find a and r and use the expression for the n^{th} term.*
48. What is the 12^{th} term of the G.P. 1, – 4, 16, – 64, …..?	*Find a and r and use the expression for the n^{th} term.*

49. Given that 2^9 is the n^{th} term of 8, 16, 32, ….. find n.	*n^{th} term $ar^{n-1} = 2^9$ you know a and r.*
50. Find the sum to 10 terms of the geometric sequence defined by $u_1 = 2$, $u_{n+1} = 3u_n$	*Write down the first few terms. Find a and r. Use formula for S_{10}*
51. Write down the n^{th} term of the G.P. 8, 4, 2, ….	*Simplify the expression for the n^{th} term using powers of 2.*
52. How many terms of the following geometric series are needed to have a sum greater than 2000? 3, 12, 48, ……..	*Set $S_n = 2000$ and find n. Need integer value of n where $S_n > 2000$* *Here you will obtain an equation where the unknown is a power - so you will need to take logs.*
53. The third and fifth terms of a geometric sequence are 32 and 512. Find the first term and the common ratio.	*Use formula for n^{th} term. Two possible values for r.*
54. Find the first three terms of a geometric sequence whose sum to n terms is given by $\frac{5}{2}(3^n - 1)$	*Use S_n formula to find S_1 and S_2 (and hence 2^{nd} term) and S_3 (and hence 3^{rd} term).*
55. A geometric progression $u_1, u_2, u_3, u_4, \ldots$ is given by $u_1 = 8$, $u_{n+1} = \frac{1}{2} u_n$ Does the sum of the series converge or diverge? (Say why.) If the series converges find the value to which it converges.	*Write down the first few terms. Find a and r. Use formula for S_∞*

56. Rita earns £12000 per year. She receives an annual pay increase of 8%.
(i) After how many years will she be earning more than £21000?
(ii) How much will she have earnt altogether by the end of the sixth year?

The binomial expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2\times1} + \frac{n(n-1)(n-2)x^3}{3\times2\times1} + \frac{n(n-1)(n-2)(n-3)x^4}{4\times3\times2\times1} + \ldots\ldots\ldots$$

This can be used for negative and fractional values of n provided $-1 < x < 1$ (or $|x| < 1$)

So $(1+2x)^{½} = (1+(2x))^{½} = 1 + \tfrac{1}{2}(2x) + \dfrac{\frac{1}{2}(\frac{1}{2}-1)(2x)^2}{2\times1} + \dfrac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(2x)^3}{3\times2\times1} + \ldots\ldots\ldots\ldots$

is valid for $-1 < 2x < 1$ so for $-½ < x < ½$

Example: Expand $(1+x)^5$ up to the term in x^3. $(1+x)^5 = 1 + 5x + \frac{5.4}{2.1}x^2 + \frac{5.4.3}{3.2.1}x^3 + \ldots\ldots$ $= 1 + 5x + 10x^2 + 10x^3 + \ldots..$	*The expansion is true for +ve whole number values of n, as well as negative and fractional values. If you go far enough you will get the whole expansion which will be true for all x.* *Use n = 5 5.4 means 5× 4 etc.* *Tidy up.*
57. Expand $(1-x)^5$ up to the term in x^3 Complete: $(1-x)^5 = (1+(-x))^5$ $= 1 + 5(-x) + \_\_\_\_\_\_ (-x)^2 + \_\_\_\_\_\_ (\quad)^3 + \ldots$ $=$	*The expansion is valid for $(1+x)^n$* *You must write down $(1+\quad)^{power}$* *It is best to rewrite with a + and a bracket around the (–x). You are then less likely to make a mistake.* *Use n = 5. Where there is an x in the formula use (–x). Use brackets as you are less likely to make a mistake with the powers.* *Tidy up.*
58. Expand $(1+2x)^8$ up to the term in x^3 Complete: $(1+2x)^8 = 1 + 8(2x) + \_\_\_\_\_\_(\quad) + \_\_\_\_\_\_(\quad) + \ldots$ $=$ $=$	*Use n = 8. Where there is an x in the formula use (2x). Use brackets as you are less likely to make a mistake with the powers.* *Tidy up.*

When n is fractional or negative (i.e. $n \notin \mathbb{N}$) you **must** state the values of x for which the expansion is valid.

Remember your work on indices. $\frac{1}{x} = x^{-1}$, $\sqrt{x} = x^{\frac{1}{2}}$, $\frac{1}{x^2} = x^{-2}$ *etc.*

Remember your fraction work e.g. $\dfrac{\frac{1}{2}\times\frac{1}{3}}{5} = \dfrac{1}{2\times3\times5} = \dfrac{1}{30}$

Example:	
$(1+x)^{-\frac{1}{3}} = 1+\left(-\tfrac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2.1}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3.2.1}x^3 + \ldots$	*$n = -⅓$. Put the n, (n – 1) etc in brackets. You are then less likely to make a mistake.*
$= 1 - \tfrac{1}{3}x + \tfrac{4}{3.3.2.1}x^2 - \tfrac{4.7}{3.3.3.3.2.1}x^3 + \ldots$	*Sort out the sign for each term then deal with the fractions. Remember your*
$= 1 - \tfrac{1}{3}x + \tfrac{2}{9}x^2 - \tfrac{14}{81}x^3 + \ldots.$	*fraction work e.g.* $\frac{\frac{1}{2}\times\frac{1}{3}}{5} = \frac{1}{2\times3\times5} = \frac{1}{30}$
Valid for $-1 < x < 1$	*Do not forget this part.*

Write the following as a series in ascending powers of x up to the term in x^3 (Completing where relevant.)

59. $(1+x)^{\frac{1}{2}} = 1 + \qquad + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2.1}x^2 + \frac{\qquad}{} \qquad +\ldots.$ Valid for	*$n = ½$. Put the n, (n – 1) etc in brackets when it goes –ve. You are then less likely to make a mistake.* *Sort out the sign for each term then deal with the fractions. Remember your fraction work e.g.* $\frac{\frac{1}{2}\times\frac{1}{3}}{5} = \frac{1}{2\times3\times5} = \frac{1}{30}$
60. $\frac{1}{(1+x)^2} = (1+x)^{-2}$ Valid for	*Remember your work on indices.* $\frac{1}{x} = x^{-1}$, $\sqrt{x} = x^{\frac{1}{2}}$, $\frac{1}{x^2} = x^{-2}$ *etc.* *$n = -2$. Put the n, (n – 1) etc in brackets. You are then less likely to make a mistake.* *Tidy up.*
61. $\sqrt[3]{1-x} =$ Valid for	*Rewrite as a bracket to a power.* *Remember you must have (1 +)^{power}* *Sort out the sign for each term then deal with the fractions.* *Tidy up.* *Here you are using (–x) for x, so –1<–x<1 Multiplying through by –1 (and changing the signs) you get 1>x>–1 which is the same as –1<x<1*

To use the expansion you need (1 +   )^{power}

Here are some examples of how you may have to rewrite an expression before you can apply the binomial expansion.

For $(1+2x)^{-3}$ rewrite as $(1+(2x))^{-3}$ and use the binomial expansion with n = – 3 and replacing x with $(2x)$.

For $(1-3x)^{-1}$ rewrite as $(1+(-3x))^{-1}$ and use the binomial expansion with n = – 1 and replacing x with $(-3x)$.

For $(1+x^2)^{\frac{1}{2}}$ rewrite as $(1+(x^2))^{\frac{1}{2}}$ and use the binomial expansion with n = ½ and replacing x with (x^2).

For $(2+x)^{-2}$ take the 2 out as a factor $(2+x)^{-2} = (2(1+\frac{x}{2}))^{-2} = 2^{-2}(1+(\frac{x}{2}))^{-2}$ and use the binomial expansion on the bracket with n = – 2 and replacing x with $(\frac{x}{2})$. Then multiply throughout by the 2^{-2}

For $(x+1)^{\frac{1}{2}}$ rewrite as $(1+x)^{\frac{1}{2}}$ and use the binomial expansion with n = ½

For $(x+2)^{\frac{1}{3}}$ rewrite as $(2+x)^{\frac{1}{3}}$ and then take 2 out as a factor and use the binomial expansion as shown above.

For $(x-1)^{-1}$ rewrite as $(-1+x)^{-1}$ then take –1 out as a factor so $(-1+x)^{-1} = (-1(1-x))^{-1} = -(1+(-x))^{-1}$ and use the expansion on the bracket with n = – 1 and replacing x with $(-x)$. Then multiply throughout by the – (or –1)

Write the following as a series in ascending powers of x up to the term in x^3 (Completing where relevant.)

62. $(1-2x)^8 = (1+(-2x))^8$ $= 1 + 8(-2x) +$ $=$	*Rewrite in the form $(1+x)^n$* *Where there is an x in the formula use (–2x). Here n = 8.* *Leave in the brackets round the (–2x)* *Careful when multiplying out the brackets. Tidy up. Valid for all x.*
63. $\sqrt{(1+2x)} = (1+(2x))^{\frac{1}{2}}$ $= 1 + \frac{1}{2}(2x) +$ $=$ valid for	*Where there is an x in the formula use (2x). Use brackets as you are less likely to make a mistake with the powers.* *Here n= ½ . Careful when multiplying out the brackets. Tidy up.* *Here you are using (2x) for x, so –1 < 2x < 1 Dividing through by 2 you get –½ < x < ½*

64. $\dfrac{1}{\sqrt{1+2x}} = (\quad\quad)$ valid for	*Write the root as a power and bring to the top as a –ve power.* *(You do **not** want $\frac{1}{1+\frac{1}{2}(2x)+.....}$ it is not in ascending powers of x.* *i.e. $1 + ax + bx^2 + ...$)* *Where there is an x in the formula use (2x). Use brackets as you are less likely to make a mistake with the powers.*
65. $\sqrt{(1+x)^3} =$ valid for	*Rewrite in the form $(1 + x)^n$*
66. $(1+5x)\sqrt{1-2x} =$ valid for	*Rewrite the second expression in the form $(1 + x)^n$.* *Then rewrite as a series.* *Then multiply by (1+5x), [Multiply each term by 1 and each term by +5x]* *Ignore terms higher than x^3.* *Collect like terms.* *(i.e. constants, x, x^2, x^3)* *Here the series was used for the second expression. So the result is valid for $-1<(-2x)<1$.* *Rewrite with $<x<$ or $\|x\|<$*
67. $(4+x)^{1/2} = \left\{4\left(1+\frac{x}{4}\right)\right\}^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$ valid for	*You need $(1 + ax)^n$ to use the expansion. So take the 4 out as a factor. (Notice you the have x/4 for x.)* *The 4 is raised to the power as well as the bracket.* *Now do the expansion on the bracket using $\left(\frac{x}{4}\right)$ for x. Then multiply all terms by 2.* *So the result is valid for $-1<\left(\frac{x}{4}\right)<1$.* *Rewrite with $<x<$ or $\|x\|<$*
68. Write $\dfrac{\sqrt{1+2x}}{(1-x)^2}$ as a single series in ascending powers of x. valid for	*Rewrite in the form* *$(1 + ax)^n(1 + bx)^m$.* *You will have to expand both expressions.* *Then multiply all terms in one by all terms in the other ignoring any results that give a power higher than 3. Be systematic: 1 times all the terms in the second ....then +½x times all the terms in the second ... and so on.* *Tidy up.* ***x has to be valid for both expansions so pick the one with the tighter restrictions.***

69. Write $\frac{1}{(2-x)^2}$ as a series in ascending powers of x. valid for	*Try this one yourself. Do not forget: you need $(1+ax)^n$ to use the expansion.*
70. (i) Write $\sqrt{1-2x}$ as a series in ascending powers of x up to the term in x^3. (ii) Using a suitable value for x in this expansion, find the value of $\sqrt{0.98}$ to four decimal places. (iii) Hence find the value of $\sqrt{2}$ to three decimal places.	*Do not forget: you need $(1+ax)^n$ to use the expansion and state the values of x for which the expansion is valid.* *(ii) Substitute your value of x into the expansion.* *(iii) You need to be able to connect $\sqrt{2}$ and $\sqrt{0.98}$. Rewrite $\sqrt{0.98}$ with a $\sqrt{2}$. Hint: $98 = 2 \times 49$*
71. Expand $\sqrt{1+x}$ as a series in ascending powers of x as far as the term in x^3. Use this series to determine the value of $\sqrt{101}$ to four decimal places.	*To use the expansion, for the second part, x must be between –1 and 1 and so writing 101 as 1 + 100 will not help.* *You will have to write 101 as 100+1 and then take 100 out as a factor. Both will be to the power ½.*

Mixed questions 2

72. (a) Write down the first five terms and the 50[th] term of the following sequence:

$x_n = 2n - 5 \quad (n = 1, 2, 3, ...)$

(b) Write down the first five terms of the following sequence:

$x_1 = 10, \quad x_{n+1} = 2x_n + 5 \quad (n = 1, 2, 3, ...)$

(c) Evaluate $\sum_{i=1}^{8} 2i - 1$

73. The n^{th} term of a sequence is $5n + 2$. Find the first three terms.

74. How many terms of the series 100 + 110 + 120 + 130 + … are needed for the sum to exceed 4200?
(Hint: Set $S_n = 4200$ and solve for n. Need integer value > n.)

75. (i) The k^{th} term of an arithmetic progression is given by $x_k = 2 + 5k$. Write down the first term and the common difference.

(ii) Find $\sum_{k=1}^{20} x_k$

76. Expand $(1 - 3x)^7$ as far as the term in x^3. Obtain an approximation for 0.997^7 to 6 decimal places by using a suitable value of x in this expansion. (Show your working.)

77. The 3^{rd} term of a geometric series is $\frac{1}{2}$ and the 6^{th} term is $\frac{1}{128}$. Find the sum to infinity.

78. Find the sum of 15 terms of the sequence 5, 10, 20, 40, ...

79. Expand $(1-x)^6(1+x)^8$ in ascending powers of x as far as the term in x^2

80. Runa saves £40 at the start of each month in a savings account. Interest is calculated at the end of each month and is added to her account. The total amount in her account at the end of n months is

$$\sum_{r=1}^{r=n} 40\times 1.005^r$$

(i) Find the total in her account after 5 months. (Give your answer to the nearest penny.)

(ii) How much will Runa have in her account after 10 years, if she continues to save £40 each month and never withdraws any money from the account?

81. A dressmaker's tape measure is rolled up into a tight spiral. The first (innermost) loop requires 2 cm of tape, the second 2.5 cm, the third 3 cm and so on. The full rolled tape has 16 complete loops. Find:

(i) the length of the outermost loop

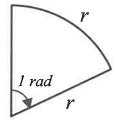
(ii) the length of the tape measure.

TRIGONOMETRY

Radians:

$$180^\circ = \pi \text{ radians}$$

Definition:
A radian is the angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



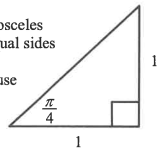
To convert degrees into radians multiply by $\frac{\pi}{180}$

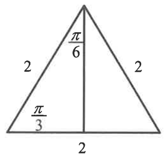
To convert radians to degrees multiply by $\frac{180}{\pi}$

Also useful to remember $30^\circ = \frac{\pi}{6}$, $60^\circ = \frac{\pi}{3}$, $45^\circ = \frac{\pi}{4}$, $90^\circ = \frac{\pi}{2}$, $360^\circ = 2\pi$

And sin, cos and tan of $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$ from simple triangles.

Right angled isosceles triangle with equal sides of length 1. Hence hypotenuse $\sqrt{2}$





Equilateral triangle of side length 2. Hence vertical height $\sqrt{3}$

82. Complete: $60^\circ = 60 \times \quad = \quad$ radians

$195^\circ = \quad \times \quad = \quad$ radians

} *Leave in terms of π*

$37^\circ = \quad \times \quad = \quad$ radians

To 3 sig. Fig.

$\frac{5\pi}{6} = \frac{5\pi}{6} \times \_ = \quad ^\circ$

$\frac{11\pi}{3} = \quad \times \quad =$

1.7 radians $= \quad \times \quad =$

To 3 sig. fig.

$\sin(\pi/4) =$

$\cos(1.2) =$

} *Calculator in radian mode.*

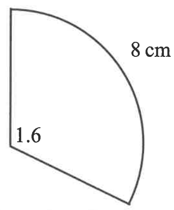
$$\text{Arc length} = r\theta \qquad \text{Area of sector} = \tfrac{1}{2} r^2 \theta \qquad \Big\}\ \theta \text{ in radians}$$

Example: An arc is of length 8 cm and it subtends an angle of 1.6 radians at the centre of the circle. Calculate the radius of the cicle.

Arc length = $r\,\theta$

$8 = 1.6\,r$

$r = 8/1.6 = 5$ cm



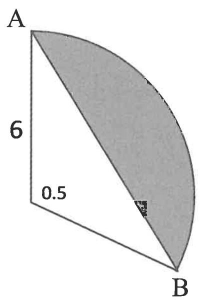
83. Find the area of the shaded segment of a circle of radius 6 cm. The angle subtended by the arc AB at the centre is 0.5 radians.

Complete:

Area of sector =
=

Area of triangle =

Area of shaded segment =



Radius 6 cm.

Use Area of a triangle = $\frac{1}{2}ab\sin C$

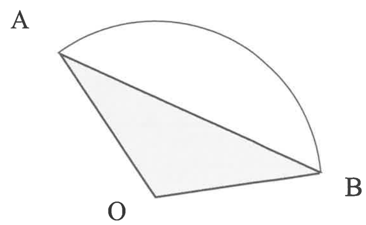
C in radians - so calculator in radian mode!

Subtract.

84. A piece of track in a child's railway set is an arc of a circle of length 30 cm and radius 100 cm. Through what angle does the direction of the track turn? (Give your answer in radians.)

85. A chord AB of length 8 cm subtends an angle of $\frac{2\pi}{3}$ at the centre of a circle.
Find (a) the length of arc AB,
(b) the area of the minor segment cut off by the chord.

86. AB is the arc of a circle centre O, radius r. Arc length AB is 66 cm. The angle subtended by the arc at the centre is 99°
(i) Find the angle in radians to 3 decimal places.
(ii) Find the area of the unshaded part of the sector shown.



You need to be able to work with radians when working with the trig ratios (sin, cos and tan). Sketch the graphs $y = \sin x$, $y = \cos x$ and $y = \tan x$ using radians for $-\pi \le x \le 2\pi$. Label $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π on the x axis and the amplitude.

87. Solve $\cos 2x = 0$ for x in the range $0 \le x \le 2\pi$ Complete: $\cos A =$ $A = \arccos \qquad =$ $A = 2x =$ $x =$	*cos of an angle (2x) equals 0* *Let the angle be A.* *Find first value for A.* *x between 0 and 2π* *Therefore A (2x) between 0 and 4π.* *Sketch cos graph.* *Work out the other values of A up to 4π.* *Divide all answers by 2 to find the values of x. Leave answers in terms of π.*
88. $\sin(x + \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ for x in the range $0 \le x \le 2\pi$ Complete: $\sin A =$ $A = \arcsin \qquad =$ $A = x + \frac{\pi}{3} =$ $x =$	*cos of an angle* $(x - \frac{\pi}{3})$ *equals* $\frac{\sqrt{3}}{2}$ *Let the angle be A.* *Find first value for A. You should recognise* $\frac{\sqrt{3}}{2}$ *as sin of one of the angles* $\frac{\pi}{6}$ *(30°),* $\frac{\pi}{3}$ *(60°) or* $\frac{\pi}{4}$ *(45°)* *x between 0 and 2π* *Therefore A* $(x+\frac{\pi}{3})$ *between* $\frac{\pi}{3}$ *and* $2\pi + \frac{\pi}{3} = \frac{7\pi}{3}$ *Sketch sin graph.* *Work out the other values of A from* $\frac{\pi}{3}$ *to* $\frac{7\pi}{3}$ *Subtract* $\frac{\pi}{3}$ *from all the answers to obtain the values of x. Leave answers in terms of π.*
89. Sketch the curve of $y = \sin(x + \frac{\pi}{4})$ for $-\pi \le x \le 2\pi$	*Label the axes. x from −π to 2π* *y from −1 to 1* *You may wish to draw y = sin x faintly to start with.* *Translate* $\frac{\pi}{4}$ *to the left.* *Easiest to mark where the curve cuts the x axis and the turning points and then join up these marks.* *Label where the curve cuts the axes and the amplitude.*

Trigonometric functions and identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Pythagorean identities

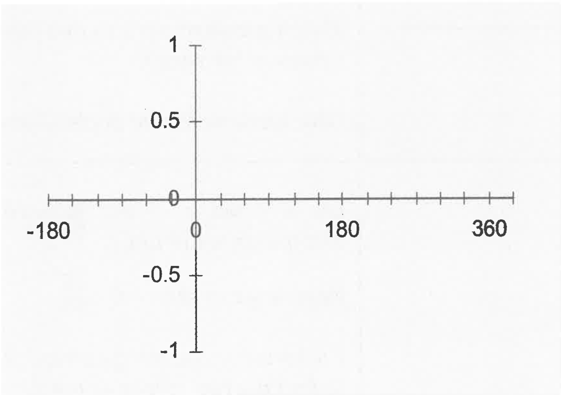
$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

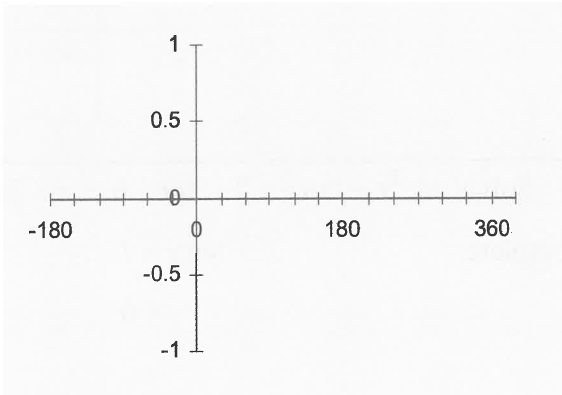
$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

Sketch the graphs (Easiest to use the sin x curve to plot the 1/sin x curve etc. or plot some points on your calculator - it will help you to remember. Do not just look it up!)

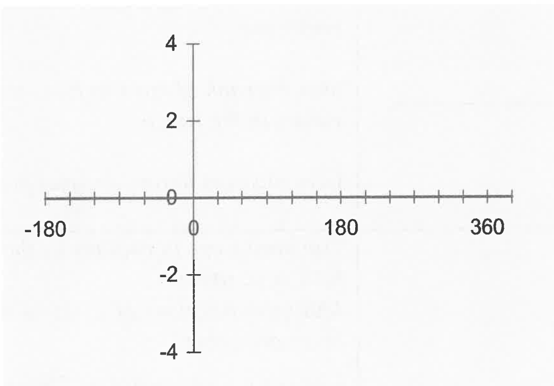
$y = \sin x$



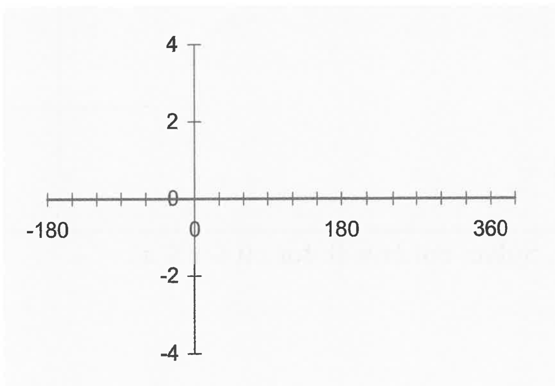
$y = \cos x$



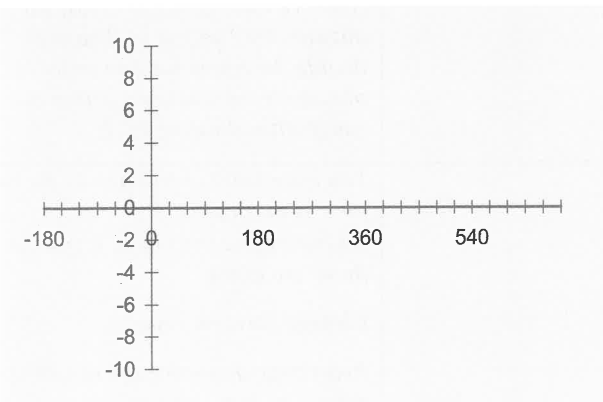
$y = \operatorname{cosec} x$



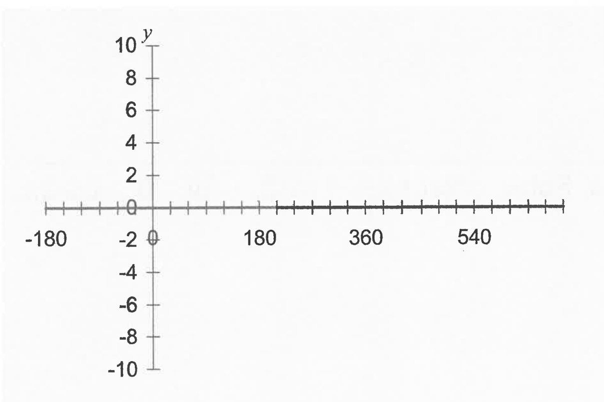
$y = \sec x$



$y = \tan x$



$y = \cot x$



Check these on your graphical calculator or a computer.

90. Complete: (a) $\sec 60^\circ =$ (b) $\cot 45^\circ =$ (c) $\operatorname{cosec} 28^\circ =$	*(a) Rewrite as $1/\cos 60^\circ$ and evaluate.* *(c) to 3 sig.fig.*
91. Solve $\sec x = 5$ for $0^\circ \leq x \leq 360^\circ$ Complete: $\overline{\qquad\qquad} = 5$ $\cos x =$ $x =$	*Put in terms of $\cos x$.* *Rearrange to give $\cos x =$* *Your calculator gives one value for x.* *Sketch graph of $\cos x$ to find other values in the range.* *Give answers to one decimal place.*
92. Solve $\sec^2 x + \tan x = 7$ for $0^\circ \leq x \leq 360^\circ$ Complete: $+ \tan x = 7$ $= 0$ $=$ $\tan x =$ or $x =$	*Use $1 + \tan^2 x = \sec^2 x$ to change to a quadratic in tan x.* *Rearrange to give = 0.* *Factorise or use the quadratic formula to find the two values of tan x.* *Your calculator gives one value for x in each case.* *Sketch graph of tan x to find other values in the range.* *Give answers to one decimal place.*
93. Solve $\cot 2x = 0$ for $0 \leq x \leq \pi$	*You must work in radians as the range for x is in radians.* *Multiples/fractions of π, if you can, as these are exact.* *Use $\cot x = \cos x/\sin x$ as $1/\tan x$ will not help you.* *Note: you are going to divide **all** your answers by 2 so you will need to double the range for x in order to obtain all the solutions within the range after dividing by 2.*
94. Solve $\operatorname{cosec}(x + \frac{\pi}{3}) = 2$ for $0 \leq x \leq 2\pi$	*You must work in radians as the range for x is in radians* *Multiples/fractions of π, if you can, as these are exact.* *Change cosec to 1/sin* *Rearrange and solve the resulting sine equation. Note: you are going to subtract $\pi/3$ from all your answers so you will need to go beyond the range for x in order to obtain all the solutions within the range after subtracting $\pi/3$.*

Further trig identities:

Given formulae.

$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$

$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$

$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Not given formulae. Know how to obtain them.

$\sin 2A \equiv 2\sin A\cos A$

$\cos 2A \equiv \cos^2 A - \sin^2 A$

$\equiv 2\cos^2 A - 1$

$\equiv 1 - 2\sin^2 A$

Which can be rearranged to give $\cos^2 A$ or $\sin^2 A$ in terms of $\cos 2A$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

Writing $2A = \theta$ so $A = \theta/2$
Then

$\cos\theta \equiv 2\cos^2\frac{\theta}{2} - 1$

and

$\cos\theta \equiv 1 - 2\sin^2\frac{\theta}{2}$

Which can be rearranged. give $\cos^2\frac{\theta}{2}$ or $\sin^2\frac{\theta}{2}$ in terms of $\cos\theta$

Difficult to remember all the formulas but easy to remember what to do to obtain them!

The Pythagorean identities and the double angle identities are often used in this and other work. You need to be familiar with them so that you recognise when it might be useful to use them. It is important that you learn them. **Cover up** the two boxes on the right above and see if you can complete these:

In terms of $\cos x$ and $\sin x$ — In terms of $\cos x$ — In terms of $\sin x$

$\cos 2x =$ $\qquad\qquad =$ $\qquad\qquad =$

Rearrange to give $\sin^2 x =$ $\qquad$ so $\sin x =$

and $\cos^2 x =$ $\qquad$ so $\cos x =$

Pythagorean identities:

$\sin^2 x +$ $\qquad =$

Rearrange to give $\sin^2 x =$

$\cos^2 x =$

Complete: $1 +$ $\qquad = \sec^2 x$

Rearrange to give $\tan^2 x =$

In terms of $\cos x$ and $\sin x$ — In terms of $\cos x$ — In terms of $\sin x$

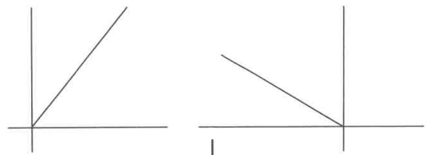
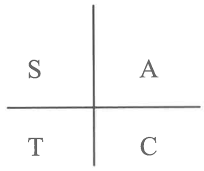
Again $\cos 2x =$ $\qquad\qquad =$ $\qquad\qquad =$

Substitute $\theta = 2x$ to obtain the half angle identities

In terms of $\cos(\theta/2)$ — In terms of $\sin(\theta/2)$

$\cos\theta =$ $\qquad$ and $\cos\theta =$

Rearrange to give $\cos^2\frac{\theta}{2} =$ $\qquad$ and $\sin^2\frac{\theta}{2} =$

95. Show that $2\cos(x + \frac{\pi}{3}) = \cos x - \sqrt{3}\sin x$	*When you see $\sqrt{3}$ or $\sqrt{2}$ in a trig question it is likely that you will need to use exact results for sin, cos and tan of $\pi/6(30^\circ)$, $\pi/3(60^\circ)$ and $\pi/4(45^\circ)$. Take the LHS and expand using cos(A+B). Show all working.*
96. Simplify $\sin(x + 30^\circ) - \sin(x - 30^\circ)$	*Rewrite each sin using the identities. Then tidy up. Careful, minus all of the identity for sin(x – 30).*
97. Prove $\cos x \cos 2x + \sin x \sin 2x = \cos x$ LHS =	*Take the Left Hand Side (LHS) and use the identities to rewrite the double angle parts. Careful cosx is multiplied by all of the alternative for cos2x. (Use cos2x in terms of sinx)*
98. If $\cos A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, where A is acute and B is obtuse, find the exact value of (i) $\cos(A + B)$ (ii) $\tan(A - B)$ (iii) $\operatorname{cosec}(A + B)$	*Work out the other trig ratios of A and B first. These may help.*  *OR*  *OR sketches of the trig graphs.*
99. Prove $\cos 3A = 4\cos^3 A - 3\cos A$ LHS =	*Take the LHS and first rewrite as cos(A + 2A). Then use the identity to rewrite.* *Then rewrite the cos2A and sin2A. You try and finish from here.*
100. Solve the equation $\cos(\theta - 60^\circ) = 3\cos\theta$ for $0^\circ \le \theta \le 360^\circ$	*Use the identity to remove the bracket. Write in the values of $\cos 60^\circ$ and $\sin 60^\circ$.* *Collect together the cos terms.* *If you have $\sin\theta$ and $\cos\theta$ in an equation, see if you can rearrange to = 0 and factorise. If this is unsuccessful look to obtain $\tan\theta$.*
101. Solve the equation $\cos\theta = \cos\frac{\theta}{2}$ for $0 \le \theta \le 2\pi$	*The clue to using the half angle identities is in the question. If you can remember the identity for cos2A try to obtain $\cos\theta$ in terms of $\cos\frac{\theta}{2}$ You should end up with a quadratic in $\cos\frac{\theta}{2}$ which you can factorise and hence find θ within the given range.*

$r\cos(\theta \pm \alpha)$, $r\sin(\theta \pm \alpha)$ method:

$a\cos\theta \pm b\sin\theta$ can be written in the form $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$ where r is positive and α is acute.

Best to show the method by example:

Write $2\cos\theta + 5\sin\theta$ in the form $r\cos(\theta - \alpha)$

$r\cos(\theta - \alpha) = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= r\cos\theta\cos\alpha + r\sin\theta\sin\alpha)$

compare with $2\cos\theta \quad + \quad 5\sin\theta$

So $r\cos\alpha = 2$

and $r\sin\alpha = 5$

Square and add

$r^2\cos^2\alpha + r^2\sin^2\alpha = 2^2 + 5^2$

$r^2(\cos^2\alpha + \sin^2\alpha) = 2^2 + 5^2 \qquad \cos^2\alpha + \sin^2\alpha = 1$

You do not have to write down these two lines.

$r^2 = 2^2 + 5^2$

$= 4 + 25$

$= 29$

$r = \sqrt{29}$

Divide the two equations

$\frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha = \frac{5}{2}$

$\alpha = 68.2^\circ$ to 1 dec. pl.

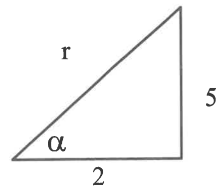
So $2\cos\theta + 5\sin\theta = \sqrt{29}\cos(\theta - 68.2^\circ)$

ALTERNATIVE METHOD

From these: $\cos\alpha = \frac{2}{r}$

and $\sin\alpha = \frac{5}{r}$

And using these we can draw a right angled triangle with angle α.



Using the triangle.

By Pythagoras $r^2 = 2^2 + 5^2$

$r = \sqrt{29}$

and $\tan\alpha = \frac{5}{2}$

$\alpha = 68.2^\circ$

This method can be used to solve equations of the form $a\cos\theta \pm b\sin\theta = c$

102. Write $5\cos x - 12\sin x$ in the form $r\cos(x + \alpha)$	*Use the identity for cos (A + B).* *Compare with 5cosx – 12sinx to work out $r\cos\alpha$ and $r\sin\alpha$* *Work out r and α and rewrite.*
103. Write $2\sin x - 3\cos x$ in the form $r\sin(x - \alpha)$	*Use the identity for sin (A – B).* *Compare with 2sinx – 3cosx to work out $r\cos\alpha$ and $r\sin\alpha$* *Work out r and α and rewrite.*
104. Write $\cos x + \sin x$ in the form $r\cos(x - \alpha)$	*Use the identity for cos (A – B).* *Compare with cosx + sinx to work out $r\cos\alpha$ and $r\sin\alpha$* *Work out r and α and rewrite.*

105. (i) Express $\sqrt{3}\sin x + \cos x$ in the form $r\sin(x + \alpha)$ where $r > 0$ and $0 < \alpha < \pi/2$

(ii) Solve $\sqrt{3}\sin x + \cos x = 1$ for $0 \le x \le 2\pi$

(iii) Find the maximum value of $\sqrt{3}\sin x + \cos x$ and the smallest value of x between 0 and 2π where this maximum value occurs.

(i) $r\sin(x + \alpha) =$

compare with $\sqrt{3}\sin x \quad + \quad \cos x$

So $\qquad = \sqrt{3}$

$\qquad = 1$

$r^2 =$ $\qquad$ so r $=$

$\tan\alpha =$ $\qquad$ so $\alpha =$

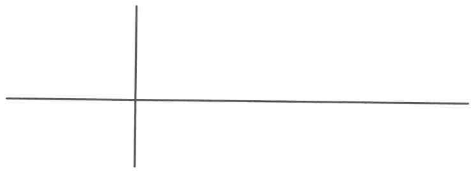
$\therefore \sqrt{3}\sin x + \cos x =$

(ii) $\sqrt{3}\sin x + \cos x = 1$

$\qquad = 1$

$\qquad =$

$(x + \quad) =$



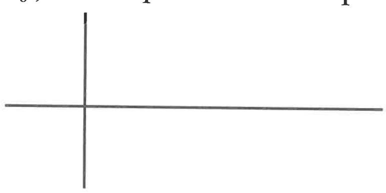
$x =$

(iii) $\sqrt{3}\sin x + \cos x =$

maximum value of $\sin(x + \frac{\pi}{6})$ is

$\therefore$ maximum value of $2\sin(x + \frac{\pi}{6})$ is

The graph of $y = 2\sin(x + \frac{\pi}{6})$ has amplitude $\qquad$ and phase shift



The graph shows a maximum of $\qquad$ and the smallest positive value of x at which it occurs is

Note: this is the form asinx + bcosx.

Use the identity to rewrite.

Compare with the expression given.

Work out r and α.

Rewrite in the form r sin (x + α) α in radians in terms of π.

Rewrite with answer obtained in part (i)

Divide to obtain sin (x +)= sin of an angle (x +) equals ½

Find the first value for the angle.

x between 0 and 2π. (x +) need a slightly different range. Sketch sine graph.

Work out the other angles that give ½ (in terms of π is best) and subtract π / 6 from each one.

(iii) Write in r sin (x + α) form.

Sketch the graph. It may help to sketch the sine curve first and then stretch by factor 2 parallel to the y axis and then shift left by π / 6.

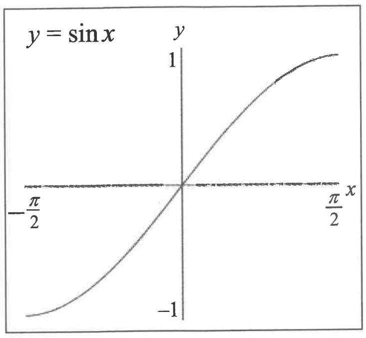
Make sure that you learn this method. This is a typical exam question.

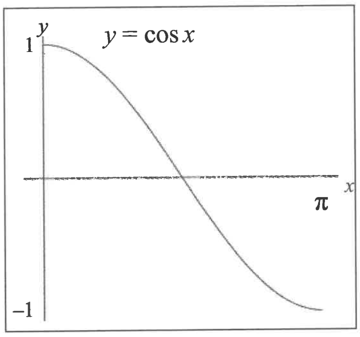
106. Express $6\cos x - 8\sin x$ in the form $r\cos(x + \alpha)$ where $r > 0$ and $0 < \alpha < \pi/2$

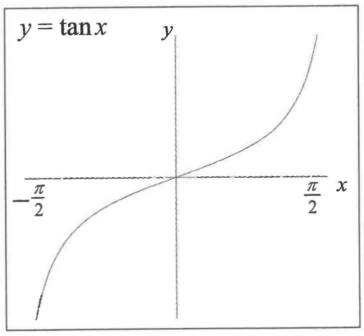
Hence solve the equation $6\cos x - 8\sin x = 9$ for $0 \le x \le 2\pi$ Give your answers to 2 decimal places.

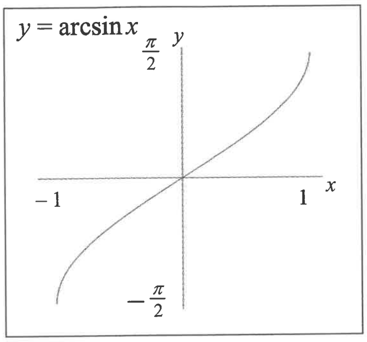
Inverse trigonometric functions:

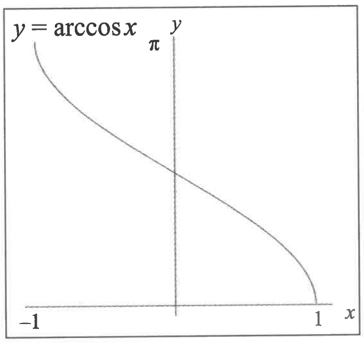
We limit the domain of the functions sine, cosine and tangent so that they are one-to-one mappings and therefore have inverse functions. Notice the domain and range of each function.
Remember the inverse functions are reflections in the line $y = x$.

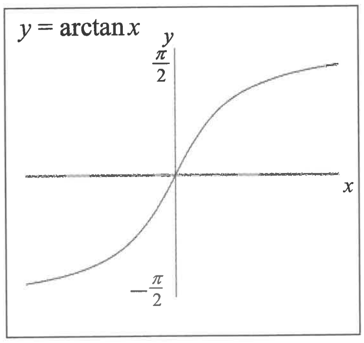












$\arctan x$ Remember this means 'the angle whose tan is x' - so it is an angle. Also written $\tan^{-1} x$
Similarly $\arcsin x$ can be written $\sin^{-1} x$ and $\arccos x$ as $\cos^{-1} x$ These are inverse functions not reciprocals.

$\tan(\arctan x)$ is just tan of an angle (here whose tan is x) so $\tan(\arctan x) = x$
Similarly $\tan(\arctan \frac{\pi}{4}) = \frac{\pi}{4}$ and $\sin(\arcsin \frac{\pi}{6}) = \frac{\pi}{6}$ etc.

107. Complete this example: Show $\arctan(2x) - \arctan(\frac{1}{x}) = \arctan(\frac{2x^2-1}{3x})$

Take the tan of both sides $\tan(\arctan(2x) - \arctan(\frac{1}{x})) = \tan(\arctan(\frac{2x^2-1}{3x}))$

RHS $= \frac{2x^2-1}{3x}$

LHS $= \tan(\arctan(2x) - \arctan(\frac{1}{x})) =$ *This is just tan (A – B) Use the formula to rewrite.*

These are just angles.

$=$ *Write without the tan (arctan) etc.*

$=$ *Tidy up. Multiply top and bottom by x.*

$=$

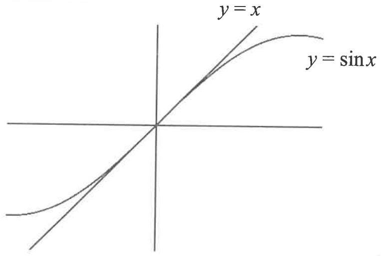
$=$ RHS *To finish.*

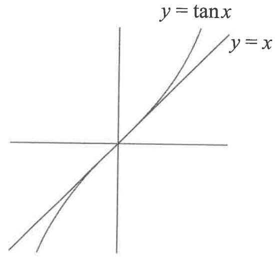
108. Show that $(1 + x)\tan(2\arctan x) = \dfrac{2x}{1-x}$

Small angle approximations of sine, cosine and tangent:

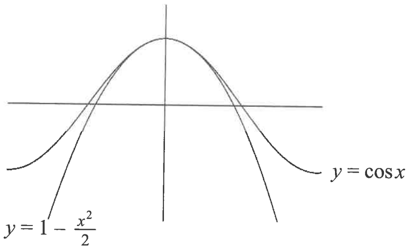
When x is **small** and in **radians**

$$\sin x \approx x \qquad \tan x \approx x \qquad \cos x \approx 1 - \frac{x^2}{2}$$





This can be shown by looking at the graphs.
Also you can check on your calculator.
$\sin 0.1 = 0.099833 = 0.100$ to 3 dec. pl.
(where 0.1 radians is about 6°)



109. Find approximations for the following when x is small:

(a) $\frac{\sin 2x}{x}$ (b) $\frac{1-\cos x}{x}$ (c) $\frac{\sin x \tan x}{x^2}$

(d) $\frac{1-\cos 2x}{x \tan x}$ (e) $\frac{x \sin 3x}{\cos 4x - 1}$

110. Find approximations for the following when θ is small:

(a) $\frac{\theta \sin \theta}{1-\cos \theta}$ (b) $\frac{\tan \theta \sin \theta}{1-\cos 2\theta}$

111. Show that if x is small $\cos\left(\frac{\pi}{3} + x\right) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{4}x^2$

112. Show that if θ is small $\frac{\sin \theta \sin(x+\theta)}{\theta} \approx \sin x$

The following are for use when differentiating $\sin x$ and $\cos x$ from first principles.

113. Find approximations for the following as $h \to 0$:

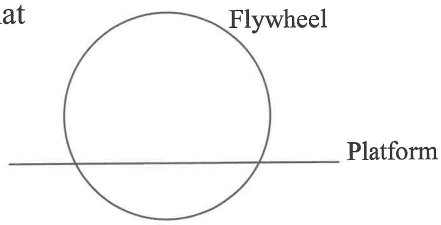
(a) $\frac{\sin h}{h}$ (b) $\frac{\cos h - 1}{h}$

Use of trigonometric functions in modelling:

114. A beam engine used for pumping water has a flywheel of diameter 6 metres. A platform, used for maintenance, is placed so that part of the wheel is above and part is below the platform. A marker is placed on the circumference of the wheel. The vertical height of the marker, h, above the platform at time, t seconds, is modelled by $h = 3\cos(\frac{2\pi}{3}t - \frac{\pi}{2}) + 1.5$

(a) What is the greatest height of the marker above the platform and at what time does this first occur?

(b) At what time is the marker first level with the platform?



115. A swing is pushed to start it swinging. The vertical height of the swing above the ground, in metres (m), at time t seconds, is modelled by $h = \frac{\sqrt{3}}{2}\cos\frac{\pi}{2}t - \frac{1}{2}\sin\frac{\pi}{2}t + 1.6$

(a) Write the model in the form $h = R\cos(\frac{\pi}{2}t + b) + c$

(b) What is the height of the swing when it starts its motion?

(c) What is the greatest height reached by the swing and when does this first occur?

(d) Why would you not use this model to predict the height after 1 minute?

Note: some your work on mechanics involves trigonometric functions.

Mixed questions 3

116. Solve the equation $\sin\theta - \cos 2\theta = 0$ for values of θ between 0 and 2π radians.

117. Express $\sin x + 3\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and α is the smallest positive angle possible. Find the maximum value of $\sin x + 3\cos x$ and state the smallest positive value of x for which it occurs.

118. Factorise $\sin 2A + \sin A$

119. Find an approximation for $\frac{\cos x - 1}{\sin x \tan x}$ if x is small.

120. Prove $\frac{1 - \cos 2x}{\sin 2x} = \tan x$

121. Express $2\cos\theta - \sin\theta$ in the form $r\cos(\theta + \alpha)$, where $r > 0$ and $0 < \alpha < \pi/2$.
Hence solve the equation $2\cos\theta - \sin\theta = 1$ for $0 \le \theta \le 2\pi$, giving your answer in radians to 2 decimal places.

DIFFERENTIATION

Differentiating e^x and $\ln x$:

Remember: $y = x^p \quad \frac{dy}{dx} = p x^{p-1}$ $y = kx^p \quad \frac{dy}{dx} = kpx^{p-1}$	$y = e^x \quad \frac{dy}{dx} = e^x$ $y = e^{ax} \quad \frac{dy}{dx} = a e^{ax}$ $y = ke^{ax} \quad \frac{dy}{dx} = k a e^{ax}$	$y = \ln x \quad \frac{dy}{dx} = \frac{1}{x}$ $y = \ln ax \quad \frac{dy}{dx} = \frac{1}{x}$ $y = k \ln ax \quad \frac{dy}{dx} = \frac{k}{x}$

122. Complete: (a) $y = 3e^x \quad \frac{dy}{dx} =$	*Use the rule.*
(b) $y = 5x^2 - e^x \quad \frac{dy}{dx} =$	*Differentiate each term. Use the rules.*
(c) $y = \ln 2x \quad \frac{dy}{dx} =$	*Use the rule.*
(d) $y = 5\ln 3x \quad \frac{dy}{dx} =$	*Use the rule.*
(e) $y = 2e^x + 3\ln x^2 =$ $\frac{dy}{dx} =$	*Use the log law to bring that power down. Then differentiate each term.*
(f) $y = \frac{2}{3}e^x - \ln\sqrt{x} =$ $\frac{dy}{dx} =$	*Rewrite the root as a power. Use the log law to bring that power down. Then differentiate each term.*
(g) $f(x) = 4e^{2x} \quad f'(x) = \quad\quad f''(x) =$	*Use the rule twice.*
(h) $y = 2\ln\frac{x}{3} \quad \frac{dy}{dx} =$	$\frac{x}{3} = \frac{1}{3} \times x$
(i) $y = 5\ln 2x^3 = \quad\quad =$ $\frac{dy}{dx} =$	*Careful: $2x^3 = 2 \times x^3$. Use the log law to split it up and then the log law to bring the power down. Then differentiate each term. 5 times them both.*
123. Find the gradient of the curve $y = 3e^{4x}$ at the point where $x = 0.5$ $\frac{dy}{dx} =$ $x = 0.5 \quad \frac{dy}{dx} =$	*Write down the 3 and differentiate the e^x using the above rule.* *Substitute in the value for x.*
124. Complete: $y = 2\sqrt{x} - \frac{3}{x^2} + 5\ln x$ $=$ $\frac{dy}{dx} =$ $=$	*Remember: Write roots as powers. For fractions with x in the denominator write the x on top with a negative power.* *Differentiate each term.* *Tidy up.*

125. Find the turning point on the curve $y = x^2 - 8\ln x$ for $x > 0$ Determine the nature of the turning point.	*Differentiate.* *Equate to 0.* *Solve for x. Don't forget $x > 0$.* *Therefore only one answer.* *Find the y coordinate.* *Determine whether max or min by finding the second differential.* *(–ve max, +ve min and 8/x is $8x^{-1}$)* *Write down the point.*
126. The population of a certain area is modelled by $P = A\,e^{0.02t}$ where P is the population at time t years and A is a constant. When t = 0 the population is 2 million. Find $\frac{dP}{dt}$. Also find the value of P when the population is growing at a rate of 0.1 million people per year.	*Working in millions.* *P = 3.4 when t = 0 gives the value of A.* *Write down the equation with this value of A.* *Differentiate to find $\frac{dP}{dt}$* *Substitute $\frac{dP}{dt} = 0.1$ to find t and hence P using the original equation with the found value of A.*

Combining functions:
It is important to be able to distinguish between products, where you will use the product rule, and composite functions (functions of functions), where you will use the chain rule.

If $f(x) = x^2$ and $g(x) = 2x + 1$

Product of functions

$$f(x) \times g(x) = x^2 \times (2x+1) = x^2(2x+1)$$

Composite functions (functions of functions)

$$fg(x) = f(2x+1) = (2x+1)^2$$

One function 'inside' another function.

127. In the following state whether you have a product(P) or a composite function(C)

(a) $(3x+5)^5$ (b) e^{3x} (c) x^2e^x

(d) $\ln 5x$ (e) $2xe^x$ (f) $e^x\ln x$

(g) $\ln x^2$ (h) $e^x(3x+1)$

Sometimes you have products or quotients that are a combination of composite functions.

e.g. $e^{3x}(5x-4)^8$ A product of composite functions.

Composite function Composite function

Chain rule:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Differentiating composite functions using the *chain rule*.

Example: $y=(2x+5)^4$ Let $u=2x+5$ then $y=u^4$ $\frac{du}{dx}=2$ and $\frac{dy}{du}=4u^3$ Using the chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ $\frac{dy}{dx}=4u^3\times 2$ $\frac{dy}{dx}=8(2x+5)^3$	*You may be able to miss out the next three rows of working. It is quite acceptable to just write down the answer by using the chain rule but you must tidy up the answer if you can.*
128. $y=(x^2-5)^5$ Find $\frac{dy}{dx}$ Complete: $u=x^2-5$ $y=u^5$ $\frac{du}{dx}=$ $\frac{dy}{du}=$ $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ $\frac{dy}{dx}=$ $=$	*Composite function.* *You may not need to put in these three lines.*
129. $y=\sqrt{(x^2-x)}$ Complete: $y=$ $u=$ $y=$ $\frac{du}{dx}=$ $\frac{dy}{du}=$ $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ $\frac{dy}{dx}=$ $=$	*Write as a bracket to a power.* *You may not need to put in these three lines.*

130. $y = e^{x^2}$ Complete: $u = x^2$ $y =$ $\frac{du}{dx} =$ $\frac{dy}{du} =$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} =$ $=$	*You may not need to put in these three lines.*

For the simpler composite functions you should be able to write down the results without writing out the chain rule in full. e.g. $y = e^{3x}$ $\frac{dy}{dx} = 3\,e^{3x}$

It is useful to remember the differential of these composite functions (or be able to 'use the chain rule in your head'):

$y = e^{ax}$	$\frac{dy}{dx} = a\,e^{ax}$	and	$y = k\,e^{ax}$	$\frac{dy}{dx} = k\,a\,e^{ax}$
$y = \ln ax$	$\frac{dy}{dx} = \frac{1}{x}$	and	$y = k \ln ax$	$\frac{dy}{dx} = \frac{k}{x}$
$y = (ax+b)^n$	$\frac{dy}{dx} = a\,n\,(ax+b)^{n-1}$	Useful one to know - **must** be linear inside the bracket.		

These ones you have used before and you should recognise them.

131. Differentiate $5\,e^{2x}$	*Write* $\frac{d(5e^{2x})}{dx} =$
132. Find $\frac{dy}{dx}$ if $y = \ln x^3$	*Composite function (or by first rewriting using the third log law).*
133. Differentiate $f(x) = \ln(3x+2)$	*Log laws are of no use here!* *Write $f'(x) =$*
134. Find $\frac{dy}{dx}$ if $y = \frac{3}{e^{2x}}$	*First try to change the expression so that you do not have a fraction.*
135. Find $\frac{dy}{dx}$ if (a) $y = (3x+2)^8$ (b) $y = \sqrt{1-x}$	*For (b) write the root as a power. Be aware of the minus.*
136. Differentiate $7\,e^{3x+2}$	

137. Differentiate $\frac{1}{x} - \frac{3}{(2x+1)} + \frac{5}{(3x-1)^2}$

Product rule:

If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ where u and v are functions of x

Differentiate the following:

138. $y = x^2(3x+4)$	*Could be differentiated by first multiplying out the bracket. But here we are practising the rule. Write down the first function, differentiate the second function plus write down the second function, differentiate the first function. (Or the other way round.) Multiply out the bracket and tidy up.*
139. $y = \sqrt{x}\,(3-x)^2 \quad =$	*Write the root as a power - just the x. The second function is a composite function. (Or you can multiply out the brackets.)* ***You may want to differentiate them separately before using the product rule.*** *Write down the first function, differentiate the second function plus write down the second function, differentiate the first function. (Or the other way round.) Tidy up.*
140. $y = 3x^3\sqrt{x+4} \quad =$	*Write the root as a power.* ***You may want to differentiate the functions separately before using the product rule.*** *Tidy up.*
141. $y = 5e^{2x}(5-x^2)$	***You may want to differentiate the functions separately before using the product rule.*** *Use the rule to differentiate $5e^{2x}$ or use the chain rule.* *Tidy up.*
142. $y = e^{2x}\ln 3x$	*Here both functions are composite functions. You may remember the rules for both rather than use the chain rule.*

Quotient rule:

$$\text{If} \quad y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

where u and v are functions of x

143. If $y = \dfrac{2x-3}{5x+1}$ find $\dfrac{dy}{dx}$	*Use the rule. In words - the bottom times the differential of the top minus the top times the differential of the bottom over the bottom squared.* *Multiply out the top and tidy up.*
144. Find $\dfrac{dy}{dx}$ when $y = \dfrac{x^2}{e^x}$	*After differentiating, factorise the top.* *Cancel the e^x. i.e. divide the top and bottom of the fraction by e^x. (This is possible as $e^x \neq 0$)*
145. $y = \dfrac{2x^3 - 3}{5x}$ Find $\dfrac{dy}{dx}$	*Multiply out the bracket.* *Tidy up.* *Divide the top and bottom by 5.*
146. $y = \dfrac{2}{3x+1}$ Find $\dfrac{dy}{dx}$	*Could be written as $2(3x+1)^{-1}$ and differentiated as a composite function. Quotient rule gives a neater form for the answer.*

Differentiating $\sin x$, $\cos x$ and $\tan x$:

Differentiating $\sin x$ from first principles.

You will have to show these are true from your small angle work if not given.

From your work on small angles (page 34) you know: As $h \to 0$, $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to 0$

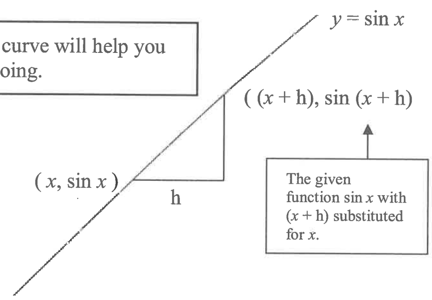
$$y = \sin x, \quad \frac{dy}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \sin x \frac{(\cos h - 1)}{h} + \cos x \frac{\sin h}{h} = \cos x$$

A sketch of part of a curve will help you 'see' what you are doing.



The given function sin x with $(x + h)$ substituted for x.

Provided that x, and therefore h, are measured in radians.

As $h \to 0$, $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to 0$

You may have been taught a slightly different proof for differentiating $\sin x$ from first principles.

147. $f(x) = \cos x$ Use differentiation from first principles to find the exact value of $f'(\frac{\pi}{3})$

You may use: As $h \to 0$, $\frac{\sin h}{h} \to 1$ *and* $\frac{\cos h - 1}{h} \to 0$

$$\frac{d(\sin x)}{dx} = \cos x \qquad \frac{d(a\sin bx)}{dx} = ab\cos bx$$

$$\frac{d(\cos x)}{dx} = -\sin x \qquad \frac{d(a\cos bx)}{dx} = -ab\sin bx$$

$$\frac{d(\tan x)}{dx} = \sec^2 x \qquad \frac{d(a\tan bx)}{dx} = ab\sec^2 bx$$

x is in radians!

148. Show that differential of $\tan x$ is $\sec^2 x$	*Only use differentiation from first principles if told to do so.* *Write tan x as sin x / cos x and differentiate as a quotient.*

Complete the following:

149. $y = 3\cos 2x$	$\frac{dy}{dx} =$	*Just use the rule above.*
150. $y = 5\sin 3x + e^x$	$\frac{dy}{dx} =$	*Just use the rules you know.*
151. $y = 4\tan 2x$	$\frac{dy}{dx} =$	*Just use the rule above.*
152. $y = \sin(x^2)$	$\frac{dy}{dx} =$	*Composite function (function of a function) - use chain rule.*
153. $y = \sin^2 x$	$\frac{dy}{dx} =$	*Rewrite as* $(\sin x)^2$ *Composite function (function of a function) - use chain rule.*
154. $y = e^x \sin 2x$	$\frac{dy}{dx} =$	*Product.*
155. $y = \frac{\cos 3x}{e^x}$	$\frac{dy}{dx} =$	*Quotient.*

156. $y = \ln(\tan x)$ $\frac{dy}{dx} =$	*Composite function (function of a function) - use chain rule.*
157. $y = \operatorname{cosec} x$ $\frac{dy}{dx} =$	*Rewrite with sinx Then write with a negative power and differentiate* ***or*** *differentiate as a quotient.*
158. $y = e^{-x}\cos 4x$ $\frac{dy}{dx} =$	*A product. Careful with the minus. Think* $e^{-x} = e^{-1x}$
159. $y = e^{\sin 2x}$ $\frac{dy}{dx} =$	*Composite function (function of a function) - use chain rule.*
160. $y = \sin(x\ln x)$ $\frac{dy}{dx} =$	*Composite function (function of a function) and product - use chain rule - and product rule.*
161. $y = \ln\frac{\sin x}{1+\cos x}$ $\frac{dy}{dx} =$	*Composite function (function of a function) and quotient - use chain rule - and quotient rule.*
162. $y = (x\tan 2x)^2$ $\frac{dy}{dx} =$	*Composite function (function of a function) and product - use chain rule - and product rule.*
163. Find the equation of the tangent to the curve $y = e^{2x}\cos x$ at the point where $x = 0$.	*Differentiate (product) and find the gradient of the tangent at x = 0.* *Use* $y - y_1 = m(x - x_1)$ *or* $y = mx + c$ *to find the equation of the tangent.*

$\frac{dy}{dx}$ and $\frac{dx}{dy}$:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Example : Find $\frac{dy}{dx}$ if $x = y^3$ $\frac{dx}{dy} = 3y^2$	*First differentiate with respect to y*
$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}$	*Use the result above to find* $\frac{dy}{dx}$
Use the original equation to write $\frac{dy}{dx}$ in terms of x if required. $y = x^{1/3}$ So $\frac{dy}{dx} = \frac{1}{3x^{2/3}}$	*You can see that this result would have been obtained by direct differentiation of* $y = x^{1/3}$

This method can be used to differentiate arcsin x, arccos x and arctan x Example: Differentiate arctan x First let $y = \arctan x$ So $\tan y = x$ or $x = \tan y$ $\frac{dx}{dy} = \sec^2 y$ $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y}$ and you know $1 + \tan^2 y = \sec^2 y$ so $\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$. $\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$	*arctan x Remember this means 'the angle whose tan is x' so y is an angle.* *Differentiate with respect to y.* *Here you need the answer in terms of x as there is no y in the original question.*
164. If $x = y^2 + 3$ find $\frac{dy}{dx}$ when $y = 2$	*First find* $\frac{dx}{dy}$ *Use the result above to find* $\frac{dy}{dx}$ *in terms of y.* *Substitute the value of y into the expression.*
165. If $x = \sin y$ find $\frac{dy}{dx}$ in terms of x	*First find* $\frac{dx}{dy}$ *Use the result above to find* $\frac{dy}{dx}$ *in terms of y. Use the identity* $\sin^2 y + \cos^2 y = 1$ *to write in terms of x knowing* $x = \sin y$ *You should state why you take the positive root. (Look at range of y and therefore range of cos x.)*
166. Make x the subject of the equation $y = \ln x$ Hence show $\frac{dy}{dx} = \frac{1}{x}$ and find $\frac{d^2y}{dx^2}$.	*Use the definition of log - here base e.* *Differentiate with respect to y.* *Use* $\frac{dy}{dx} = \frac{1}{dx/dy}$ *and* $x = e^y$ *to write* $\frac{dy}{dx}$ *in terms of x.* *Differentiate* $\frac{dy}{dx}$ *with respect to x to obtain* $\frac{d^2y}{dx^2}$ as $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ Note: $\frac{d^2y}{dx^2} \neq \frac{1}{\frac{d^2x}{dy^2}}$

Mixed questions 4:

Complete:

167. $y = 2x^3 - 5x^2 + 3x - 1$ $\frac{dy}{dx} =$

168. $y = e^{2x}$ $\frac{dy}{dx} =$

169. $y = \ln 4x$ $\frac{dy}{dx} =$

170. $y = 5e^{3x}$ $\frac{dy}{dx} =$

171. $y = 3\ln 2x$ $\frac{dy}{dx} =$

172. $y = \frac{5}{x}$ $\frac{dy}{dx} =$

173. $y = \sqrt{x^3}$ $\frac{dy}{dx} =$

174. $y = (2x - 3)^5$ $\frac{dy}{dx} =$

175. $y = \cos 4x$ $\frac{dy}{dx} =$

176. $y = \sin x^3$ $\frac{dy}{dx} =$

177. $y = \sqrt{x} + \frac{3}{x^2} + 2e^{4x} - \ln 3x$ $\frac{dy}{dx} =$

178. $y = e^{2x} \sin 2x$ $\frac{dy}{dx} =$

179. $y = \sin^2(2x)$ $\frac{dy}{dx} =$

180. $y = \frac{e^{3x}}{x^2}$ $\frac{dy}{dx} =$

181. Differentiate $f(x) = \tan^2 2x$

Remember: $\tan^2 2x$ means $(\tan 2x)^2$

182. Show that $\frac{d(\sec x)}{dx} = \sec x \tan x$

183. Differentiate $f(x) = \frac{\ln x}{x^2}$

184. If $y = e^{-x} \cos 2x$ find $\frac{dy}{dx}$

185. If $x = \tan 2y$ find $\frac{dy}{dx}$ in terms of x

Mixed questions 5

186. Find $\frac{dy}{dx}$ if $y = \sin(\sin x)$

187. If $f(x) = x^2$ sketch the graph $y = f(x)$
(a) On the same diagram sketch and label the curve $y = f(2x+4)$
(b) Write down an expression for $f(2x+4)$ in terms of x

188. The arc length of a sector of a circle of diameter 60 cm is 36 cm. What is the area of the sector?

189. Show that $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$ explaining why you take the positive root of $1 - x^2$

190. If $x = y^2 - 2$, $y > 0$, find $\frac{dy}{dx}$ in terms of x

191. Find a formula for sum of n terms of the series $2, -1, -4, -7, \ldots$

192. Solve the equation $\sin 2x = \cos x$ for x in the range $0 \le x \le 2\pi$

193. Find where the curve $y = 2e^{\frac{1}{3}x} - 4$ cuts the coordinate axes and hence sketch the curve. Find the value of x where the gradient of the curve is ⅓ in the form $a \ln b$ where a and b are rational integers.

194. An athlete is building up his fitness. On the first day he does 30 situps, and each day he increases the number of situps by 12.
(i) Find the number of situps which the athlete does on the twentieth day.
(ii) Show that the total number of situps which the athlete has done by the end of the n^{th} day is $6n^2 + 24n$
(iii) After how many days has the athlete managed 600 situps in total?

195. Show that $\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$

196. If $f(x) = 2x - 3$, $g(x) = 2e^x$ and $h(x) = 5\ln 2x$ find:
(i) $f'(x)$
(ii) $g'(x)$
(iii) $h'(x)$
(iv) $fg(x)$
(v) $gf(x)$
(vi) $hf(x)$
(vii) $f^{-1}(x)$
(viii) $g^{-1}(x)$
(ix) $h^{-1}(x)$

197. A radar scanner rotates at 20 revolutions per minute. What is this in radians per second?

198. If $y = 2\ln(3x+1)$, find the value of $\frac{dy}{dx}$ when $x = 2$

Hence find the equation of the tangent to the curve $y = 2\ln(3x+1)$ where $x = 2$

199. (a) In an arithmetic progression, the 8^{th} term is three times the 5^{th} term, and the 12^{th} term is 34. Find the first term, the common difference and the sum of the terms from the 4^{th} to the 14^{th} term inclusive.
(b) The sum, S_k of the first k terms of an arithmetic progression is $3k^2 - k$. Write down the first three terms and find a formula for the n^{th} term.

200. Show that $\arctan x + \arctan\left(\frac{1-x}{1+x}\right) = \arctan 1$

201. Differentiate the following with respect to x
(a) $\ln(x^3)$ (b) $x^3 \sin 2x$ (c) e^{5x} (d) $(5x+2)^9$

202. Prove the identity $\cos\left(\frac{\pi}{3} - A\right) - \cos\left(\frac{\pi}{3} + A\right) = \sqrt{3}\sin A$

203. Give a reason why the series $(\log_{10}2) + (\log_{10}2)^2 + (\log_{10}2)^3 + \ldots\ldots$ is convergent.
Find the sum to 10 terms and the sum to infinity (give answers to 5 decimal places).

204. Expand $(1 - 2x)^7$ in ascending powers of x up to the term in x^3

205. (i) Evaluate

$$\sum_{r=1}^{r=12} 2\left(\frac{3}{4}\right)^r$$

(ii) Find the value of

$$\sum_{r=1}^{r=\infty} 2\left(\frac{3}{4}\right)^r$$

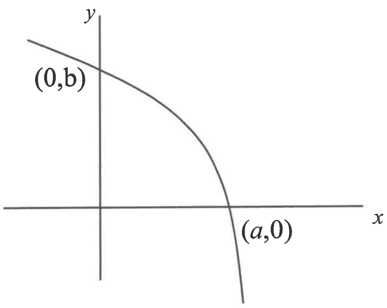
206. Solve the equation $\cot^2 x = 3\operatorname{cosec} x - 3$ for x in the range $0 \le x \le 2\pi$

207. Write $\dfrac{1}{(2-x)}$ as a series in ascending powers of x as far as the term in x^3

208. Sketch $y = \cos(2x - \pi)$ for $0 \le x \le \pi$ showing where it cuts the x and y axes.

209. (a) State the range of values for x which satisfy the inequality $|3x - 2| < 4$
(b) Solve $|2x - 3| = x + 2$

210. The diagram shows a sketch of $y = \mathrm{f}(x)$. On a separate diagram sketch $y = 2\,\mathrm{f}(\frac{1}{3}x)$



211. (a) Express $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x + \alpha)$ where $r > 0$ and $0 < \alpha < \pi/2$
(b) Solve the equation $\cos x - \sqrt{3}\sin x = 1$ for $0 \le x \le 2\pi$

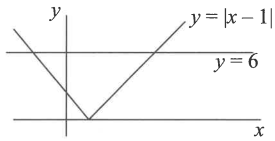
212. If θ is small find an approximation for $\frac{\sin 2\theta \tan\theta}{1 - \cos 2\theta}$

213. Differentiate the following with respect to x
(a) $\frac{2}{\sqrt{3x-1}}$ (b) $\frac{x^2-3}{5x}$ (c) $\frac{1}{x} + \ln 2x - \sqrt{x} + 2e^{4x}$

Answers

Page3

1.
$x - 1 = 6$
$x = 7$
$-x + 1 = 6$
$x = -5$
$\therefore\ x = -5$ or 7



2. $2x - 1 = x + 2$
$x = 3$
and $-(2x - 1) = x + 2$
$-2x + 1 = x + 2$
$x = -1/3$
$\therefore\ x = -1/3$ or 3

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3. $2x + 3 = -x \quad \therefore\ x = -1$
$-2x - 3 = -x \quad \therefore\ x = -3$
Solution $-3 < x < -1$

4. $x - 5 = 3 \quad \therefore\ x = 8$
$-x + 5 = 3 \quad \therefore\ x = 2$
Solution $x < 2$ or $x > 8$

5. Sketch shows you need intersections with $-x + 1$
$-x + 1 = 2x + 3 \quad \therefore\ x = -2/3$
$-x + 1 = -2x - 3 \quad \therefore\ x = -4$
Solution $x \le -4$ or $x \ge -2/3$

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6. (i) $fg(5) = f(6) = 36$ (ii) $fg(-2) = f(-8) = 64$
(iii) $fg(x) = f(2x - 4) = (2x - 4)^2$
(iv) $gf(5) = g(25) = 46$ (v) $gf(-2) = g(4) = 4$
(vi) $gf(x) = g(x^2) = 2x^2 - 4$

7. (i) $h(2) = 8$ (ii) $gh(2) = g(8) = 64 - 1 = 63$
(iii) $fgh(2) = f(63) = \ln 63$ (iv) $fgh(-2) = \ln 15$
(v) $gh(x) = g(3x + 2) = (3x + 2)^2 - 1$
$= 9x^2 + 12x + 3$
(vi) $fgh(x) = \ln(9x^2 + 12x + 3)$

8. (i) $f(3) = 9$ (ii) $g(3) = 2$ (iii) $fg(3) = f(2) = 5$
(iv) $gf(3) = g(9) = -4$
(v) $fg(x) = 4(5 - x) - 3 = 17 - 4x$
(iv) $gf(x) = 5 - (4x - 3) = 8 - 4x$

9. $x > 3/4$

10. $\frac{1}{2} < x \le 5$

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11. $4x/3 = y - 2$ *or* $4x = 3y - 6$
$y = 4x/3 + 2 \qquad y = (4x + 6)/3$
$f^{-1}(x) = \frac{4x}{3} + 2$ or $\frac{4x+6}{3}$

12. $x(y + 3) = 1$
$y + 3 = 1/x$
$y = 1/x - 3$
$f^{-1}(x) = \frac{1}{x} - 3$ or $\frac{1-3x}{x}$

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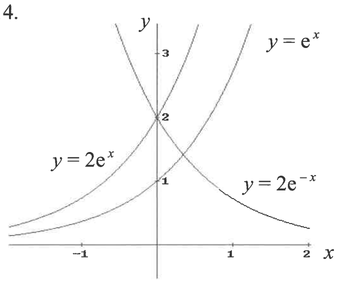
13. $y = x + 1 / x - 1$
$x = y + 1 / y - 1$
$x(y - 1) = y + 1$
$xy - x = y + 1$
$xy - y = x + 1$
$y(x - 1) = x + 1$
$y = \frac{x+1}{x-1} \qquad f^{-1}(x) = \frac{x+1}{x-1}$

14.



15. (i) (2, 25) (ii) (5/3, 19) (iii) (14, 6)

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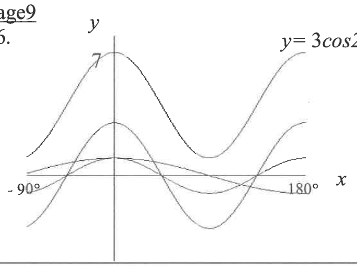
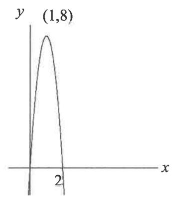
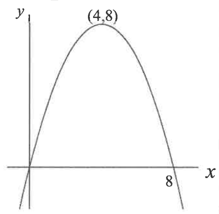
16.

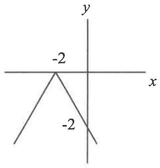


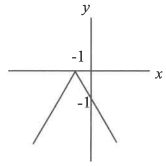
17. $y = 2f(2x) \qquad y = 2f(\frac{1}{2}x)$



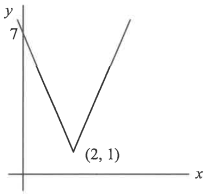


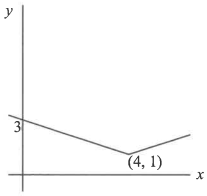
18. $y = -f(x + 3) \qquad y = -f(-x)$





19. $y = 3f(x) + 1 \qquad y = f(\frac{1}{2}x) + 1$





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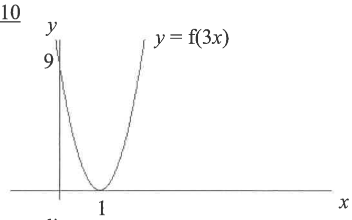
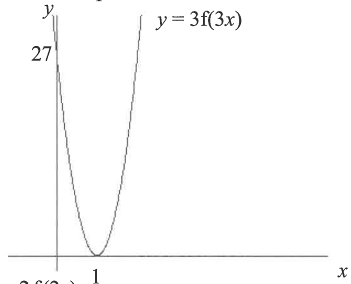
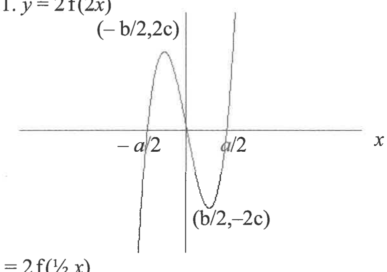
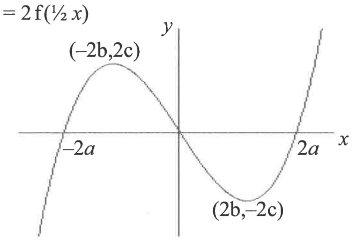




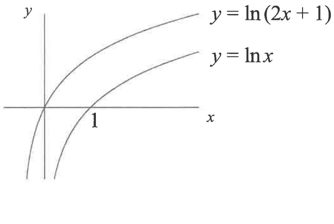
21. $y = 2f(2x)$



$y = 2f(\frac{1}{2}x)$



22.



23. $y = D/x \quad$ D = distance
$500 = D/10 \quad \therefore\ D = 5000$ miles
$600 = 5000/x\ \therefore\ x = 5000/600 = 8$ hours 20 min
Take off and landing not taken into consideration. Tailwind and/or headwind not taken into consideration.

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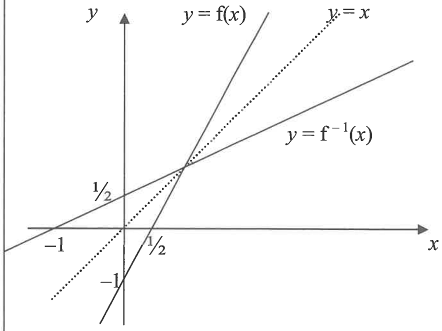
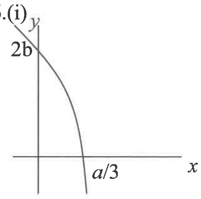
24. $N \propto 1/n \Rightarrow N = k/n \Rightarrow 120 = k/4 \Rightarrow k = 480$
$30 = 480 / n \Rightarrow n = 160$
Not everybody works at the same rate.

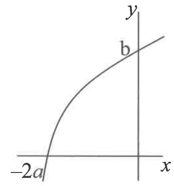
25. $f^{-1}(x) = \frac{x+1}{2} = \frac{1}{2}x + \frac{1}{2}$



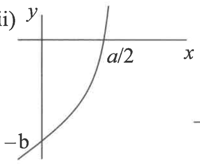
26.(i)



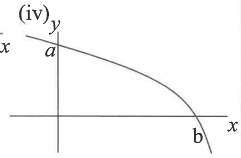
(ii)



(iii)



(iv)



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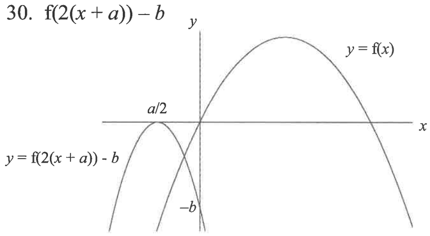
27. (i) $f(3) = 9$ (ii) $fg(2) = 64$ (iii) $fgh(1) = 529$
(iv) $fg(x) = (5x + 2)^2$
(v) $gh(x) = 5(2x + 3) - 2 = 10x + 13$

28. Reflection in $y = x$ since these functions are inverses.

29. (a) (i) Reflection in x axis
(ii) Reflection in $y = x$
(iii) Reflection in y axis
(b) y axis is a line of symmetry

30. $f(2(x + a)) - b$



31. $fg(x) = 3x^2 - 1 \qquad gf(x) = (3x - 1)^2$
$ff(x) = 3(3x - 1) - 1 = 9x - 4$

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32.
$$fg(x) = \frac{2\frac{1}{x}}{\left(\frac{1}{x}\right)^2 + 1} = \frac{2/x}{1/x^2 + 1} = \frac{2x}{1 + x^2} = f(x)$$

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33. (a) Increasing (b) Periodic
(c) Decreasing (d) Decreasing

34. (a) 7, 12, 17, 22, 27 — 20th term 102
(b) 1, 2, 4, 8, 16 — 20th term $2^{19} = 524288$
(c) −1, 4, −9, 16, −25 — 20th term 400

35. (a) 100, 85, 70, 55, 40
(b) 1, 3/2, 7/4, 15/8, 31/16
(c) 10, 25, 55, 115, 235

36. (a) $2 + 5 + 8 + 11 + 14 = 40$
(b) $3 + 5 + 9 + 17 = 34$
(c) $9 + 16 + 25 + 36 + 49 + 64 = 199$

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37. $2 + 14 \times 7 = 100$

38. $a = 1$, $d = 2$, $S_{25} = 12.5\,(2 + 24 \times 2) = 625$

39. $2 + (n-1)6 = 50$
$6(n-1) = 48$
$n - 1 = 8$
$n = 9$

40. $4 + 3d = 31 \Rightarrow 3d = 27 \Rightarrow d = 9$
$p = 13,\ q = 22$

41. $a + 2d = 17$ and $a + 6d = 5$
Subtracting $4d = -12 \Rightarrow d = -3$
Subst. $a - 6 = 17 \Rightarrow a = 23$

42. $S_1 = 5$, $S_2 = 12$ So 2^{nd} term is 7.
$S_3 = 21$, So 3^{rd} term is 9

43. $S_{30} = 30/2\,(30 + 1) = 15 \times 31 = 465$

44. $S_{100 \text{ to } 200} = S_{200} - S_{99}$
$= 200/2\,(200 + 1) - 99/2\,(99 + 1)$
$= 100 \times 201 - 99 \times 50 = 15150$

45. $S_{25} = 25/2(25 + 1) = 25 \times 13 = 325$

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46. $S_{60} - S_{39} = 60/2(60 + 1) - 39/2(39 + 1)$
$= 30 \times 61 - 39 \times 20 = 1830 - 780 = 1050$

47. $1 \times 5^{14} = 6\,103\,515\,625$

48. $1 \times (-4)^{11} = -4\,194\,304$

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49. $2^9 = 8 \times 2^{n-1} \Rightarrow 64 = 2^{n-1} \Rightarrow n - 1 = 6 \Rightarrow n = 7$

50. $a = 2, r = 3 \quad S_{10} = \dfrac{2(1-3^{10})}{1-3} = 59048$

51. $8 \times (½)^{n-1} = 2^3 / 2^{n-1} = 2^{3-(n-1)} = 2^{4-n}$

52. $2000 = 3(4^n - 1)/3 \Rightarrow 2000 = 4^n - 1 \Rightarrow 2001 = 4^n$
$\log 2001 = n \log 4 \Rightarrow n = \log 2001 / \log 4$
$n = 5.48$ take $n = 6$.

53. $512/52 = r^2 \Rightarrow r = \pm 4$, $a = 2$

54. $S_1 = 5$, $S_2 = 20$ so 2nd term is 15, $S_3 = 65$,
3^{rd} term is 45

55. $r = ½$ which is < 1 therefore it will converge.
$S_\infty = 8/½ = 16$

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56. (i) $21000 = 12000 \times 1.08^{n-1} \Rightarrow 1.75 = 1.08^{n-1}$
$\Rightarrow \log 1.75 = (n-1)\log 1.08$
$n = 8.27141$ i.e. 9^{th} year
(ii) $S_6 = 12000(1.08^6 - 1)/1.08 - 1 = £88\,031.12$

57. $(1 - x)^5 = (1 + (-x))^5$
$= 1 + 5(-x) + \frac{5.4}{2.1}(-x)^2 + \frac{5.4.3}{3.2.1}(-x)^3 + \ldots$
$= 1 - 5x + 10x^2 - 10x^3 + \ldots$

58. $(1 + 2x)^8 = 1 + 8(2x) + \frac{8.7}{2.1}(2x)^2 + \frac{8.7.6}{3.2.1}(2x)^3 +.$
$= 1 + 16x + 112x^2 + 448x^3 + \ldots$

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59. $(1 + x)^{½} = 1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2.1}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}x^3$
$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - ..$
valid for $-1 < x < 1$

60. $(1 + x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2.1}x^2 + \frac{(-2)(-3)(-4)}{3.2.1}x^3 + ..$
$= 1 - 2x + 3x^2 - 4x^3 + ..$
valid for $-1 < x < 1$

61.
$= 1 + (\frac{1}{3})(-x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2.1}(-x)^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3.2.1}(-x)^3$
$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 - \ldots.$
valid for $-1 < x < 1$

62. $(1 - 2x)^8 = (1 + (-2x))^8$
$= 1 + 8(-2x) + \frac{8.7}{2.1}(-2x)^2 + \frac{8.7.6}{3.2.1}(-2x)^3$
$= 1 - 16x + 112x^2 - 448x^3 + ..$

63. $(1 + 2x)^{½} = (1 + (2x))^{½}$
$= 1 + \frac{1}{2}(2x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2.1}(2x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}(2x)^3 +.$
$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \ldots.$
valid for $-1 < 2x < 1$ i.e. $-½ < x < ½$

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64.
$(1 + 2x)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2.1}(2x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3.2.1}(2x)^3 + \ldots$
$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + ..$
valid for $-1 < 2x < 1$ i.e. $-½ < x < ½$

65. $(1+x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{(\frac{3}{2})(\frac{1}{2})}{2.1}x^2 + \frac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})}{3.2.1}x^3 + ..$
$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + ..$
valid for $-1 < x < 1$

66. $(1 + 5x)(1 + (-2x))^{\frac{1}{2}}$
$= (1+5x)\left(1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2.1}(-2x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}(-2x)^3 + ..\right)$
$= (1+5x)(1 - x - \frac{x^2}{2} - \frac{x^3}{2} - ..) = 1 - x - \frac{x^2}{2} - \frac{x^3}{2} - .. + 5x - 5x^2 - \frac{5x^3}{2} - ..$
$= 1 + 4x - \frac{11}{2}x^2 - 3x^3 - ..$
valid for $-1 < -2x < 1$ i.e. $-½ < x < ½$

67.
$(1 + (2x))^{\frac{1}{2}} = 1 + \frac{1}{2}(2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2.1}(2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}(2x)^3 + ..$
$= 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - ..$
$(1 + (-x))^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2.1}(-x)^2 + \frac{(-2)(-3)(-4)}{3.2.1}(-x)^3$
$= 1 + 2x + 3x^2 + 4x^3 + ..$
$(1+2x)^{\frac{1}{2}}(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + x + 2x^2 + 3x^3 - \frac{x^2}{2} - x^3 + \frac{x^3}{2}$
$= 1 + 3x + \frac{9}{2}x^2 + \frac{13}{2}x^3 \ldots.$
valid for $-1 < 2x < 1$ and $-1 < -x < 1$
i.e. for $-½ < x < ½$

68. $2\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$
$= 4^{\frac{1}{2}}\left(1 + \frac{1}{2}\left(\frac{x}{4}\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2.1}\left(\frac{x}{4}\right)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}\left(\frac{x}{4}\right)^3 + ..\right)$
$= 2(1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} + ..) = 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512}$
valid for $-1 < x/4 < 1$ i.e. $-4 < x < 4$

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69. $(2 - x)^{-2} = \left\{2\left(1 - \frac{x}{2}\right)\right\}^{-2} = 2^{-2}\left(1 + \left(-\frac{x}{2}\right)\right)^{-2}$
$= \frac{1}{4}\left\{1 + (-2)\left(-\frac{x}{2}\right) + \frac{(-2)(-3)}{2.1}\left(-\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3.2.1}\left(-\frac{x}{2}\right)^3 + ..\right\}$
$= \frac{1}{4}\left\{1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + ..\right\} = \frac{1}{4} + \frac{x}{4} + \frac{3}{16}x^2 + \frac{x^3}{8} + ..$
valid for $-1 < -x/2 < 1$ i.e. $-2 < x < 2$

70.(i) $(1 + (-2x))^{\frac{1}{2}}$
$= 1 + \frac{1}{2}(-2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2.1}(-2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}(-2x)^3 + ..$
$= 1 - x - \frac{x^2}{2} - \frac{x^3}{2} - ..$
valid for $-1 < -2x < 1$ i.e. $-½ < x < ½$

(ii) $x = 0.01$ $\quad \sqrt{0.98} = 1 - 0.01 - \frac{0.0001}{2} - \frac{0.000001}{2}$
$= 0.9899$

(iii) $\sqrt{0.98} = \sqrt{\frac{98}{100}} = \sqrt{\frac{49 \times 2}{100}} = \frac{7}{10}\sqrt{2}$
$\sqrt{2} = \frac{10}{7} \times 0.9899 = 1.414$

71. $(1 + x)^{½} = 1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2.1}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}x^3$
$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - ..$
valid for $-1 < x < 1$
$\sqrt{101} = \sqrt{100 + 1} = \sqrt{100\left(1 + \frac{1}{100}\right)} = 10(1 + 0.01)^{\frac{1}{2}}$
$= 10\left(1 + \frac{0.01}{2} - \frac{0.0001}{8} + \frac{0.000001}{16} - ..\right)$
$= 10.04987563 = 10.0499$

72. (a) $-3, -1, 1, 3, 5$ $\quad 50^{th}$ term is 95
(b) 10, 25, 55, 115, 235
(c) $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64$

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73. 7, 12, 17

74. $S_n = n/2(200 + (n-1)10)$
$= n(200 + 10n - 10)/2$
$100n + 5n^2 - 5n = 4200$
$5n^2 + 95n = 4200$
$5n^2 + 95n - 4200 = 0$
$n^2 + 19n - 840 = 0$
$(n + 40)(n - 21) = 0$
$n = -40$ not possible, $n = 21$
Sum to 22 terms will exceed 4200

75. (i) $a = 7$, $d = 5$ (ii) $S_{20} = 10(14 + 19 \times 5) = 1090$

76. $(1 - 3x)^7 = 1 + 7(-3x) + \frac{7.6}{2.1}(-3x)^2$
$+ \frac{7.6.5}{3.2.1}(-3x)^3$
$= 1 - 21x + 189x^2 - 945x^3$
$0.997^7 = 1 - 21(0.001) + 189(.001)^2 - 945(.001)^3$
$= 1 - 0.021 + 0.000189 - 0.000000945$
$= 0.979188$ to 6 d.p.

77. $\frac{ar^5}{ar^2} = \frac{\frac{1}{128}}{\frac{1}{2}} = \frac{1}{64}$
$\therefore r = ¼, \quad a = 8 \quad S_\infty = \frac{8}{1 - \frac{1}{4}} = \frac{32}{3}$

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78. $S_{15} = 5(1 - 2^{15}) / (1 - 2) = 163835$

79. $(1 - x)^6 = (1 + 6(-x) + \frac{6.5}{2.1}(-x)^2 + ..)$
$= 1 - 6x + 15x^2$
$(1 + x)^8 = (1 + 8x + \frac{8.7}{2.1}x^2 + \ldots) = 1 + 8x + 28x^2$
$(1 - x)^6(1 + x)^8 = (1 - 6x + 15x^2)(1 + 8x + 28x^2)$
$= 1 + 8x + 28x^2 - 6x - 48x^2 + 15x^2 + \ldots$
$= 1 + 2x - 5x^2$

80. (i) $a = 40.2$, $r = 1.005$
$S_5 = 40.2(1.005^5 - 1)/(1.005 - 1) = £203.02$
(ii) $S_{120} = £6587.95$

81. $a = 2$, $d = 0.5$ last loop $2 + 15d = 2 + 7.5$
$= 9.5$ cm
length of tape $= 16(2 + 9.5) / 2$
$= 8 \times 11.5 = 92$ cm

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82. $60° = 60 \times \pi/180 = \pi/3$ radians
$195° = 195 \times \pi/180 = 13\pi/12$ radians
$37° = 37 \times \pi/180 = 0.646$ radians
$\frac{5\pi}{6} = \frac{5\pi}{6} \times \frac{180}{\pi} = 150°$
$\frac{11\pi}{3} = \frac{11\pi}{3} \times \frac{180}{\pi} = 660°$
1.7 radians $= 1.7 \times 180/\pi = 97.4°$
$\sin(\pi/4) = 0.707$
$\cos(1.2) = 0.362$

83. Area of sector $= ½\, 6^2 \times 0.5 = 9$
Area of triangle $= ½\, 6 \times 6 \times \sin 0.5 = 8.630$
Area of shaded segment $= 9 - 8.630 = 0.370$

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84. $30 = 10\theta \quad \theta = 30/10 = 0.3$ radians

85. $\sin(\pi/3) = 4/r \quad r = \frac{4}{\sin\frac{\pi}{3}} = 8/\sqrt{3}$
Arc AB $= r\,\theta = 9.67$
Area $= ½\, r^2\theta - ½\, r.r\sin\theta$
$= (½)(64/3)(2\pi/3) - (½)(64/3)\sin(2\pi/3)$
$= 22.34 - 9.2376 = 13.1$ sq.units

86. $\theta = 99 \times \pi \div 180 = 1.727875959 = 1.728$ radians
$66 = r\,\theta$
$r = 66 \div \theta = 66 \div 1.727875959$
$= 38.19718635$
Area of sector $= 0.5 \times 38.19..^2 \times 1.727875959$
$= 1260.507$
Area of triangle $= 0.5 \times 38.19\ldots \times 38.19.. \sin 1.7278$
$= 720.531$
Unshaded area $= 1260.507 - 720.531$
$= 539.976 = 540$ cm^2 to 3 sig.fig.

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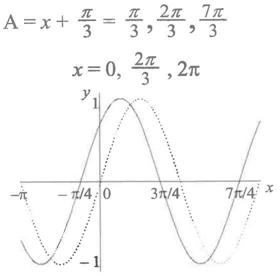
87. $\cos A = 0$
$A = 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$
$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

88. $\sin A = \sqrt{3} / 2$
$A = x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$
$x = 0, \frac{2\pi}{3}, 2\pi$

89.


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90\. (a) $\sec 60 = \frac{1}{\cos 60} = \frac{1}{½} = 2$

(b) $\cot 45 = \frac{1}{\tan 45} = \frac{1}{1} = 1$

(c) $\operatorname{cosec} 28 = \frac{1}{\sin 28} = \frac{1}{0.46947} = 2.13$

91\. $\frac{1}{\cos\theta} = 5$

$\cos x = 1/5$

$x = 78.5^\circ, 281.5^\circ$

92\. $\tan^2 x + 1 + \tan x = 7$

$\tan^2 x + \tan x - 6 = 0$

$(\tan x + 3)(\tan x - 2) = 0$

$\tan x = 2$ or -3

$x = 63.4^\circ, 108.4^\circ, 243.4^\circ, 288.4^\circ$

93\. $\frac{\cos 2x}{\sin 2x} = 0$

$\cos 2x = 0$

$2x = \pi/2, 3\pi/2$ so $x = \pi/4, 3\pi/4$

94\. $\frac{1}{\sin(x+\frac{\pi}{3})} = 2$

$\sin(x + \frac{\pi}{3}) = ½$

$(x + \frac{\pi}{3}) = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$

$x = \frac{3\pi}{6}, \frac{11\pi}{6} = \frac{\pi}{2}, \frac{11\pi}{6}$

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95\. LHS $= 2\cos x\cos\frac{\pi}{3} - 2\sin x\sin\frac{\pi}{3}$

$= 2.\cos x\,\frac{1}{2} - 2\sin x\frac{\sqrt{3}}{2}$

$= \cos x - \sqrt{3}\sin x =$ RHS

96\. $\sin x\cos 30 + \cos x\sin 30 - (\sin x\cos 30 - \cos x\sin 30)$

$= 2\cos x\sin 30 = 2\cos x\,\frac{1}{2} = \cos x$

97\. LHS $= \cos x(1 - 2\sin^2 x) + \sin x 2\sin x\cos x$

$= \cos x - 2\sin^2 x\cos x + 2\sin^2 x\cos x = \cos x$

98\. $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$

$\sin B = \frac{5}{13}$, $\cos B = -\frac{12}{13}$, $\tan B = -\frac{5}{12}$

$\cos(A + B) = \frac{3}{5}(-\frac{12}{13}) - \frac{4}{5}\times\frac{5}{13} = -\frac{56}{65}$

$\tan(A - B) = \frac{\frac{4}{3}-(-\frac{5}{12})}{1-\frac{4}{3}(-\frac{5}{12})} = \frac{\frac{16+5}{12}}{1+\frac{20}{36}} = \frac{21}{12}\div\frac{16}{56} = \frac{21}{12}\times\frac{56}{16} = \frac{9}{8}$

$\operatorname{cosec}(A + B) = \frac{1}{\frac{4}{5}(-\frac{12}{13})+\frac{3}{5}\times\frac{5}{13}} = \frac{1}{\frac{15-48}{65}} = -\frac{65}{33}$

99\. LHS $= \cos(A + 2A) = \cos A\cos 2A - \sin A\sin 2A$

$= \cos A(2\cos^2 A - 1) - \sin A 2\sin A\cos A$

$= 2\cos^3 A - \cos A - 2\sin^2 A\cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A =$ RHS

100\. $\cos\theta\cos 60 + \sin\theta\sin 60 = 3\cos\theta$

$\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = 3\cos\theta$

$\frac{\sqrt{3}}{2}\sin\theta = \frac{5}{2}\cos\theta$

$\frac{\sin\theta}{\cos\theta} = \frac{5}{2}\times\frac{2}{\sqrt{3}} = \frac{5}{\sqrt{3}}$

$\theta = 70.9^\circ, 250.9^\circ$

101\. $2\cos^2(\theta/2) - 1 = \cos(\theta/2)$

$2\cos^2(\theta/2) - \cos(\theta/2) - 1 = 0$

$(2\cos(\theta/2) + 1)(\cos(\theta/2) - 1) = 0$

$\cos(\theta/2) = 1$ or $-½$

$\theta/2 = 0, 2\pi/3$ $\therefore$ $\theta = 0, 4\pi/3$

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102\. $r\cos x\cos\alpha - r\sin x\sin\alpha$

$r\cos\alpha = 5$

$r\sin\alpha = 12$

$r = \sqrt{(5^2 + 12^2)} = 13$

$\tan\alpha = 12/5$ $\alpha = 67.4^\circ$

$5\cos x - 12\sin x = 13\cos(x + 67.4^\circ)$

103\. $r\sin x\cos\alpha - r\cos x\sin\alpha$

$r\cos\alpha = 2$

$r\sin\alpha = 3$

$r = \sqrt{4+9} = \sqrt{13}$

$\tan\alpha = 3/2$ $\alpha = 56.3^\circ$

$2\sin x - 3\cos x = \sqrt{13}\sin(x - 56.3^\circ)$

104\. $r\cos x\cos\alpha + r\sin x\sin\alpha$

$r\cos\alpha = 1$ and $r\sin\alpha = 1$

$r = \sqrt{(1 + 1)} = \sqrt{2}$

$\tan\alpha = 1$ $\therefore$ $\alpha = \pi/4$

$\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$

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105\. (i) $r\sin(x + \alpha) = r\sin x\cos\alpha + r\cos x\sin\alpha$

$\sqrt{3}\sin x + \cos x$

So $\sqrt{3} = r\cos\alpha$ and $1 = r\sin\alpha$

$r^2 = (\sqrt{3})^2 + 1^2 = 3 + 1$ so $r = 2$

$\tan\alpha = 1/\sqrt{3}$ so $\alpha = \pi/6$

$\sqrt{3}\sin x + \cos x = 2\sin(x + \pi/6)$

(ii) $\sqrt{3}\sin x + \cos x = 1$

$2\sin(x + \pi/6) = 1$

$\sin(x + \pi/6) = ½$

$(x + \pi/6) = \pi/6, 5\pi/6, 13\pi/6$

$x = 0, 2\pi/3, 2\pi$

(iii) $\sqrt{3}\sin x + \cos x = 2\sin(x + \pi/6)$

maximum value of $\sin(x + \pi/6)$ is 1

maximum value of $2\sin(x + \pi/6)$ is 2

The graph of $y = 2\sin(x + \frac{\pi}{6})$ has amplitude 2 and phase shift $-\pi/6$

The graph shows a maximum of 2 and the smallest positive value of x at which it occurs is $\pi/3$

106\. $r\cos x\cos\alpha - r\sin x\sin\alpha$

$r\cos\alpha = 6$ and $r\sin\alpha = 8$

$r = \sqrt{(36 + 64)} = 10$

$\tan\alpha = 8/6$ $\alpha = 0.927$

$6\cos x - 8\sin x = 10\cos(x + 0.927) = 9$

$\cos(x + 0.927) = 0.9$

$(x + 0.927) = 0.451, 5.83, 6.734$

$x = 4.90, 5.81$ radians

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107\. RHS $= \frac{2x-1}{3}$

LHS $= \tan(\arctan(2x) - \arctan(\frac{1}{x}))$

$= \frac{2x-\frac{1}{x}}{1+2x\frac{1}{x}} = \frac{\frac{2x^2-1}{x}}{3} = \frac{2x^2-1}{3x} =$ RHS

108\. LHS $= \frac{(1 + x)\,2\tan(\arctan x)}{(1 - \tan^2(\arctan x))}$

$= (1 + x)\,2x / (1 - x^2)$

$= (1 + x)\,2x / (1 - x)(1 + x)$

$= 2x / (1 - x) =$ RHS

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109\. (a) $\frac{2x}{x} = 2$ (b) $\frac{1-(1-x^2/2)}{x} = \frac{x}{2}$

(c) $\frac{x\times x}{x} = 1$ (d) $\frac{1-(1-4x^2/2)}{x\times x} = \frac{4x^2}{2x^2} = 2$

(e) $\frac{x\times 3x}{1-\frac{4x^2}{2}-1} = -\frac{3x^2}{8x^2} = -\frac{3}{8}$

110\. (a) $\frac{\theta\times\theta}{1-(1-\frac{\theta^2}{2})} = 2$ (b) $\frac{\theta\times\theta}{1-(1-\frac{4\theta^2}{2})} = \frac{1}{2}$

111\. $\cos\frac{\pi}{3}\cos x - \sin\frac{\pi}{3}\sin x$

$= \frac{1}{2}(1 - \frac{x^2}{2}) - \frac{\sqrt{3}}{2}x = \frac{1}{2} - \frac{x^2}{4} - \frac{\sqrt{3}}{2}x$

$= \frac{1}{4}(2 - 2\sqrt{3}\,x - x^2)$

112\. $\frac{\theta(\sin x\cos\theta + \cos x\sin\theta)}{\theta}$

$= \sin x(1 - \frac{\theta^2}{2}) + \theta\cos x$

$= \sin x - \frac{\theta^2}{2}\sin x + \theta\cos x$

$\approx \sin x$

113\. (a) $\frac{h}{h} = 1$ (b) $\frac{1-h^2/2-1}{h} = -\frac{h}{2} \to 0$ as $h \to 0$

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114\. (a) Max when cos is 1 $\therefore$ $\frac{2\pi}{3}t + \frac{\pi}{2} = 0$

$t = ¾$ sec

Max height above platform = 3 + 1.5 = 4.5 m

(b) h = 0 $\cos(\frac{2\pi}{3}t - \frac{\pi}{2}) = -\frac{1}{2}$

$\frac{2\pi}{3}t - \frac{\pi}{2} = \frac{2\pi}{3}$ $\Rightarrow$ $t = 7/4$ sec

115\. (a) $R = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$ $\tan b = \frac{½}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

$b = \pi/6$

$\therefore$ $h = \cos(\frac{\pi}{2}t + \frac{\pi}{6}) + 1.6$

(b) $t = 0$, $h = \cos\frac{\pi}{6} + 1.6 = 2.47$ m to 2 dec. pl.

(c) Max when $\cos(\frac{\pi}{2}t + \frac{\pi}{6}) = 1$

Max height = 2.6 m

Occurs when $\frac{\pi}{2}t + \frac{\pi}{6} = 0, 2\pi, ...$

$t = -1/3, 11/3, ..$

$\therefore$ Max height occurs after $3\,1/3$ sec.

(d) Motion is damped.

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116\. $2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = ½$ or -1

$\theta = \pi/6, 5\pi/6$ or $3\pi/2$

117\. $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = 1$ $R\sin\alpha = 3$

$R = \sqrt{9+1} = \sqrt{10}$ $\tan\alpha = 3/1$ $\alpha = 71.6^\circ$

$\therefore$ $\sin x + 3\cos x = \sqrt{10}\sin(x + 71.6^\circ)$

Max value is $\sqrt{10}$

Occurs $\sin(x + 71.6^\circ) = 1$

$(x + 71.6^\circ) = 90^\circ$ $\Rightarrow$ $x = 18.4^\circ$

118\. $2\sin A\cos A + \sin A = \sin A(2\cos A + 1)$

119\. $\frac{(1-x^2/2)-1}{x\times x} = \frac{-x^2/2}{x^2} = \frac{-x^2}{2x^2} = -\frac{1}{2}$

120\. LHS $= \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \frac{2\sin^2 x}{2\sin x\cos x}$

$= \frac{\sin x}{\cos x} = \tan x$

121\. $r\cos\alpha = 2$ $r\sin\alpha = 1$

$r = \sqrt{4+1} = \sqrt{5}$ $\tan\alpha = ½$ $\alpha = 0.4636$

$2\cos\theta - \sin\theta = \sqrt{5}\cos(\theta + 0.46)$

$\sqrt{5}\cos(\theta + 0.46) = 1$

$\cos(\theta + 0.46) = \frac{1}{\sqrt{5}}$

$(\theta + 0.46) = 1.1071, 5.1760$

$\theta = 0.64, 4.71$

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122\. (a) $\frac{dy}{dx} = 3e^x$ (b) $\frac{dy}{dx} = 10x - e^x$

(c) $\frac{dy}{dx} = \frac{1}{x}$ (d) $\frac{dy}{dx} = 5\times\frac{1}{x} = \frac{5}{x}$

(e) $y = 2e^x + 3\ln x^2 = 2e^x + 3\times 2\ln x$

$\frac{dy}{dx} = 2e^x + 6\times\frac{1}{x} = 2e^x + \frac{6}{x}$

(f) $y = \frac{2}{3}e^x - \ln\sqrt{x} = \frac{2}{3}e^x - \frac{1}{2}\ln x$

$\frac{dy}{dx} = \frac{2}{3}e^x - \frac{1}{2}\times\frac{1}{x} = \frac{2}{3}e^x - \frac{1}{2x}$

(g) $f(x) = 4e^{2x}$ $f'(x) = 8e^{2x}$ $f''(x) = 16e^{2x}$

(h) $y = 2\ln\frac{x}{3}$ $\frac{dy}{dx} = 2\times\frac{1}{x} = \frac{2}{x}$

(i) $y = 5\ln 2x^3 = 5(\ln 2 + \ln x^3) = 5\ln 2 + 15\ln x$

$\frac{dy}{dx} = 15\times\frac{1}{x} = \frac{15}{x}$

123\. $\frac{dy}{dx} = 12e^{4x}$ $x = 0.5$, $\frac{dy}{dx} = 88.7$ to 3 s.f.

124\. $y = 2x^{½} - 3x^{-2} + 5\ln x$

$\frac{dy}{dx} = x^{-½} + 6x^{-3} + \frac{5}{x} = \frac{1}{\sqrt{x}} + \frac{6}{x^3} + \frac{5}{x}$

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125\. $\frac{dy}{dx} = 2x - \frac{8}{x}$

For turning point $2x - \frac{8}{x} = 0$

$2x^2 - 8 = 0$

$x^2 = 4$

$x = 2$ (or $x = -2$ not possible as $x > 0$)

$y = 4 - 8\ln 2$

$\frac{d^2y}{dx^2} = 2 - -8x^{-2} = 2 + \frac{8}{x^2}$

$x = 2$, $\frac{d^2y}{dx^2} = 4$ i.e. +ve

$\therefore$ a minimum point at $(2, 4 - 8\ln 2)$

126. $t = 0$, $P = 2$ ∴ $A = 2$

$P = 2e^{0.02t}$

$\frac{dP}{dt} = 0.04\,e^{0.02t}$

$0.1 = 0.04\,e^{0.02t}$

$e^{0.02t} = 2.5$

$0.02t = \ln 2.5$

$t = \ln 2.5 / 0.02$

$P = 2e^{0.02 \times \ln 2.5 / 0.02} = 5$ million

127. (a) C (b) C (c) P
(d) C (e) P (f) P
(g) C (h) P

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128. $\frac{du}{dx} = 2x \qquad \frac{dy}{du} = 5u^4$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10xu^4 = 10x(x^2-5)^4$

129. $y = (x^2 - x)^{½}$

$u = x^2 - x \qquad y = u^{1/2}$

$\frac{du}{dx} = 2x - 1 \qquad \frac{dy}{du} = ½u^{-1/2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2x-1)\,½\,u^{-1/2} = \frac{2x-1}{2\sqrt{x^2-x}}$

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130. $u = x^2 \qquad y = e^u$

$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2xe^u = 2xe^{x^2}$

131. $10e^{2x}$

132. $\frac{dy}{dx} = 3x^2\,\frac{1}{u} = \frac{3x^2}{x^3} = \frac{3}{x}$

133. $f'(x) = \frac{3}{3x+2}$

134. $\frac{dy}{dx} = -6e^{-2x} = -\frac{6}{e^{2x}}$

135. (a) $\frac{dy}{dx} = 8(3x+2)^7 \times 3 = 24(3x+2)^7$

(b) $\frac{dy}{dx} = ½(1-x)^{-½} \times (-1) = -½(1-x)^{-½}$

136. $\frac{dy}{dx} = 21\,e^{3x+2}$

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137. $\frac{dy}{dx} = -x^{-2} - 3(-1)(2x+1)^{-2} \times 2 + 5(-2)(3x-\ )^{-3} \times 3$

$= -\frac{1}{x^2} + \frac{6}{(2x+1)^2} - \frac{30}{(3x-1)^3}$

138. $\frac{dy}{dx} = x^2(3) + (3x+4)2x$

$= 3x^2 + 6x^2 + 8x = 9x^2 + 8x$

139. $y = \sqrt{x}\,(3-x)^2 = x^{½}(3-x)^2$

$\frac{dy}{dx} = x^{½}2(3-x)(-1) + (3-x)^2\,½x^{-½}$

$= 2\sqrt{x}(x-3) + \frac{1}{2\sqrt{x}}(3-x)^2$

140. $y = 3x^3(x+4)^{½}$

$\frac{dy}{dx} = 3x^3\,\frac{1}{2}(x+4)^{-\frac{1}{2}} + 9x^2(x+4)^{\frac{1}{2}}$

$= \frac{3x^3}{2\sqrt{x+4}} + 9x^2\sqrt{x+4}$

141. $\frac{dy}{dx} = 5e^{2x}(-2x) + (5-x^2)\,10e^{2x}$

$= -10x\,e^{2x} + 10\,e^{2x}(5-x^2)$

142. $\frac{dy}{dx} = e^{2x} \times \frac{1}{x} + \ln 3x \times 2e^{2x} = \frac{e^{2x}}{x} + 2e^{2x}\ln 3x$

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143. $\frac{dy}{dx} = \frac{(5x+1)2 - (2x-3)5}{(5x+1)^2}$

$= \frac{10x + 2 - 10x + 15}{(5x+1)^2} = \frac{17}{(5x+1)^2}$

144. $\frac{dy}{dx} = \frac{e^x 2x - x^2 e^x}{e^{2x}} = \frac{2x - x^2}{e^x}$

145. $\frac{dy}{dx} = \frac{5x(6x^2) - (2x^3-3)5}{25x^2}$

$= \frac{30x^3 - 10x^3 + 15}{25x^2} = \frac{4x^3+3}{5x^2}$

146. $\frac{dy}{dx} = \frac{(3x+1)0 - 2.3}{(3x+1)} = \frac{-6}{(3x+1)^2}$

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147. $f'(\frac{\pi}{3}) = \lim_{h\to 0} \frac{\cos(\pi/3+h) - \cos(\pi/3)}{h}$

$= \lim_{h\to 0} \frac{\cos(\pi/3)\cosh - \sin(\pi/3)\sinh - 1/2}{h}$

$= \lim_{h\to 0} \frac{(1/2)\cosh - (\sqrt{3}/2)\sinh - 1/2}{h}$

$= \lim_{h\to 0} \frac{(1/2)(\cosh - 1)}{h} - \frac{(\sqrt{3}/2)\sinh}{h}$

$= -\frac{\sqrt{3}}{2}$

148. $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$= \sec^2 x$

149. $\frac{dy}{dx} = -3\sin 2x \times 2 = -6\sin 2x$

150. $\frac{dy}{dx} = 5 \times \cos 3x \times 3 + e^x = 15\cos 3x + e^x$

151. $\frac{dy}{dx} = 4\sec^2 2x \times 2 = 8\sec^2 2x$

152. $\frac{dy}{dx} = \cos x^2 \times 2x = 2x\cos(x^2)$

153. $y = (\sin x)^2$, $\frac{dy}{dx} = 2\sin x\cos x$ ($= \sin 2x$)

154. $\frac{dy}{dx} = 2e^x\cos 2x + e^x\sin 2x$

$= e^x(2\cos 2x + \sin 2x)$

155. $\frac{dy}{dx} = \frac{-3e^x\sin 3x - e^x\cos 3x}{(e^x)^2} = \frac{-3\sin 3x - \cos 3x}{e^x}$

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156. $\frac{dy}{dx} = \frac{1}{\tan x}\sec^2 x = \frac{\cos x}{\sin x\cos^2 x} = \frac{1}{\sin x\cos x}$

($= \sec x \operatorname{cosec} x$)

157. $y = (\sin x)^{-1}$

$\frac{dy}{dx} = -(\sin x)^{-2}\cos x = \frac{-\cos x}{\sin^2 x}$ ($= -\cot x \operatorname{cosec} x$)

158. $\frac{dy}{dx} = -e^{-x}\sin 4x \times 4 + \cos 4x \times e^{-x}(-1)$

$= -4e^{-x}\sin 4x - e^{-x}\cos 4x$

159. $\frac{dy}{dx} = 2e^{\sin 2x}\cos 2x$

160. $\frac{dy}{dx} = \cos(x\ln x) \times (x\frac{1}{x} + \ln x)$

$= \cos(x\ln x) + \ln x\cos(x\ln x)$

$= \cos(x\ln x)[1 + \ln x]$

161. $\frac{dy}{dx} = \frac{1}{\frac{\sin x}{1+\cos x}}\left\{\frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}\right\}$

$= \frac{(1+\cos x)(\cos x + \cos^2 x + \sin^2 x)}{\sin x(1+\cos x)^2}$

$= \frac{(1+\cos x)(\cos x + 1)}{\sin x(1+\cos x)^2} = \frac{1}{\sin x}$

162. $\frac{dy}{dx} = 2(x\tan x)(x\,2\sec^2 2x + \tan 2x)$

$= 4x^2\tan 2x\sec^2 2x + 2x\tan^2 2x$

163. $\frac{dy}{dx} = e^{2x}\sin x + 2\cos x\,e^{2x}$

$x = 0 \quad \frac{dy}{dx} = 0 + 2 = 2$ and $y = 1$

$y - 1 = 2(x - 0)$

$y = 2x + 1$

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164. $\frac{dx}{dy} = 2y \qquad \frac{dy}{dx} = \frac{1}{2y}$

$y = 2 \qquad \frac{dy}{dx} = \frac{1}{4}$

165. $\frac{dx}{dy} = \cos y$

$\cos y = \sqrt{1-\sin^2 y}$

$y = \arcsin x$ has range $-\pi/2 \le y \le \pi/2$
So for this range for y, $\cos y \ge 0$ and so take the positive root.

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

166. $x = e^y$

$\frac{dx}{dy} = e^y \qquad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\frac{1}{x^2}$

167. $\frac{dy}{dx} = 6x^2 - 10x + 3$ 168. $\frac{dy}{dx} = 2e^{2x}$

169. $\frac{dy}{dx} = \frac{1}{4x} \times 4 = \frac{1}{x}$ 170. $\frac{dy}{dx} = 5 \times 3e^{3x} = 15e^{3x}$

171. $\frac{dy}{dx} = 3 \times \frac{1}{2x} \times 2 = \frac{3}{x}$

172. $y = 5x^{-1} \quad \frac{dy}{dx} = -5x^{-2} = -\frac{5}{x^2}$

173. $y = x^{3/2} \quad \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

174. $\frac{dy}{dx} = 5(2x-3)^4 \times 2 = 10(2x-3)^4$

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175. $\frac{dy}{dx} = -4\sin 4x$

176. $\frac{dy}{dx} = 3x^2\cos x^3$

177. $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 3(-2)x^{-3} + 2 \times 4e^{4x} - \frac{1}{3x} \times 3$

$= \frac{1}{2\sqrt{x}} - \frac{6}{x^3} + 8e^{4x} - \frac{1}{x}$

178. $\frac{dy}{dx} = e^{2x} \times 2\cos 2x + \sin 2x \times 2e^{2x}$

$= 2e^{2x}(\cos 2x + \sin 2x)$

179. $\frac{dy}{dx} = 2\sin 2x \times 2\cos 2x = 4\sin 2x\cos 2x$

($= 2\sin 4x$)

180. $\frac{dy}{dx} = \frac{x^2 3e^{3x} - e^{3x}2x}{x^4} = \frac{3xe^{3x} - 2e^{3x}}{x^3} = \frac{e^{3x}(3x-2)}{x^3}$

181. $f'(x) = 2\tan 2x \times 2\sec^2 2x = 4\tan 2x\sec^2 2x$

182. $\frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x \times 0 - -\sin x}{\cos^2 x} = \frac{\sin x}{\cos x\cos x}$

$= \tan x\sec x$

183. $f'(x) = \frac{x^2 \times \frac{1}{x} - \ln x \times 2x}{x^4} = \frac{1 - 2\ln x}{x^3}$

184. $\frac{dy}{dx} = -e^{-x} \times (-2\sin 2x) + \cos 2x \times e^{-x}(-1)$

$= -e^{-x}(2\sin 2x + \cos 2x)$

185. $\frac{dx}{dy} = 2\sec^2 2y \qquad 1 + \tan^2 2y = \sec^2 2y$

$= 2 + 2\tan^2 2y = 2 + 2x^2$

$\frac{dy}{dx} = \frac{1}{2+2x^2}$

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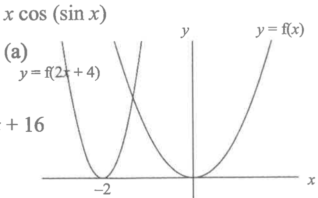
186. $\frac{dy}{dx} = \cos x\cos(\sin x)$

187. (a)



(b) $f(2x+4)$
$= 4x^2 + 16x + 16$

188. $36 = 30 \times \theta$ ∴ $\theta = 1.2$

Area $= 0.5 \times 30^2 \times 1.2 = 540$ cm^2

189. $y = \arccos x$ so $x = \cos y$ $\frac{dx}{dy} = -\sin y$

$\sin y = \sqrt{1-\cos^2 y}$

$y = \arccos x$ has range $0 \le y \le \pi$
So for this range for y, $\sin y \ge 0$ and so take the positive root.

$\frac{dy}{dx} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$

190. $y = \sqrt{x-2}$ positive as $y > 0$ given.

$\frac{dx}{dy} = 2y \qquad \frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x+2}}$

191. $S_n = n(4 + (n-1)(-3))/2 = n(4 - 3n + 3)/2$
$= n(7 - 3n)/2$

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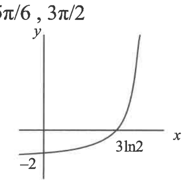
192. $2\sin x \cos x - \cos x = 0$
$\cos x\,(2\sin x - 1) = 0$
$\cos x = 0 \Rightarrow x = \pi/2,\ 3\pi/2$
or $2\sin x - 1 = 0$
$\sin x = ½ \Rightarrow x = \pi/6,\ 5\pi/6$
$x = \pi/6, \pi/2, 5\pi/6, 3\pi/2$

193. $x = 0, y = 2 - 4 = -2$
$y = 0,\ 0 = 2e^{1/3x} - 4$
$4 = 2e^{1/3x}$
$2 = e^{\frac{1}{3}x}$
$\ln 2 = ⅓ x$
$3\ln 2 = x$



$\frac{dy}{dx} = \frac{2}{3}e^{\frac{1}{3}x} \qquad so\ \frac{1}{3} = \frac{2}{3}e^{\frac{1}{3}x}$

$\frac{1}{2} = e^{\frac{1}{3}x}$

$\ln½ = ⅓ x$ (note: $\ln ½ = \ln 1 - \ln 2 = -\ln 2$)
$-\ln 2 = ⅓ x \Rightarrow x = -3\ln 2$

194. (i) $30 + 19 \times 12 = 258$

(ii) $S_n = \frac{n}{2}\{2x30 + (n-1)12\} = 30n + 6n^2 - 6n$
$= 6n^2 + 24n$

(iii) $600 = 6n^2 + 24n$
$6n^2 + 24n - 600 = 0$
$n = 8.198$ (or -12.198 not possible)
Total achieved on the 9^{th} day

195. LHS $= \frac{\sin x + 2\sin x\cos x}{1 + \cos x + (2\cos^2 x - 1)} = \frac{\sin x(1 + 2\cos x)}{\cos x + 2\cos^2 x}$

$= \frac{\sin x(1 + 2\cos x)}{\cos x(1 + 2\cos x)} = \frac{\sin x}{\cos x} = \tan x$

196. (i) 2
(ii) $2e^x$
(iii) $5/x$
(iv) $2(2e^x) - 3 = 4e^x - 3$
(v) $2e^{2x-3}$
(vi) $5\ln 2(2x - 3) = 5\ln(4x - 6)$
(vii) $y = 2x - 3$ Interchange x and y
$x = 2y - 3$
$y = \frac{x+3}{2} \quad \therefore\ f^{-1}(x) = \frac{x+3}{2}$

(viii) $y = 2e^x$ Interchange x and y
$x = 2e^y$
$\frac{x}{2} = e^y$
$\ln\frac{x}{2} = y\ln e = y \quad \therefore\ g^{-1}(x) = \ln\frac{x}{2}$

(ix) $y = 5\ln 2x$ Interchange x and y
$x = 5\ln 2y$
$\frac{x}{5} = \ln 2y,\ \therefore e^{x/5} = 2y$
so $h^{-1}(x) = \frac{1}{2}e^{\frac{x}{5}}$

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197. $20 \times 2\pi$ radians in 60 secs
$40\pi/60$ radians per sec $= {}^2/_3\ \pi$ radians per sec

198. $\frac{dy}{dx} = 2\times\frac{1}{3x+1}\times 3 = \frac{6}{3x+1}$

$x = 2,\ y = 2\ln 7,\ \frac{dy}{dx} = \frac{6}{7}$

equation of tangent $y - 2\ln 7 = \frac{6}{7}(x - 2)$
$7y - 14\ln 7 = 6x - 12$
$7y - 6x = 14\ln 7 - 12$

199. (a) $a + 7d = 3 \times (a + 4d) \Rightarrow 2a + 5d = 0$
$a + 11d = 34$
solve simultaneously $a = -10, d = 4$
$S_{14} - S_3 = 242$
(b) 2, 8, 14 $\quad 2 + (n-1)6 = 6n - 4$

200. tan of LHS $= \frac{x + \frac{1-x}{1+x}}{1 - x(\frac{1-x}{1+x})} = \frac{x(1+x) + (1-x)}{(1+x) - x(1-x)}$

$= \frac{x + x^2 + 1 - x}{1 + x - x + x^2} = \frac{x^2 + 1}{1 + x^2} = 1 =$ tan of RHS

201. (a) $\frac{dy}{dx} = \frac{1}{x^3}\times 3x^2 = \frac{3}{x}$

(b) $\frac{dy}{dx} = x^3 2\cos 2x + \sin 2x \times 3x^2$
$= x^2(2x\cos 2x + 3\sin 2x)$

(c) $\frac{dy}{dx} = 5e^{5x}$

(d) $\frac{dy}{dx} = 45(5x + 2)^8$

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202. LHS $= \cos\pi/3\cos A + \sin\pi/3\sin A$
$-(\cos\pi/3\cos A - \sin\pi/3\sin A)$
$= 2\sin\pi/3\sin A = 2\frac{\sqrt{3}}{2}\sin A = \sqrt{3}\sin A$

203. Geometric series $a = 2$, $r = \log_{10} 2 = 0.301..$
$\therefore$ as $r < 1$ it will converge

$S_{10} = \frac{\log 2(1 - \log 2^{10})}{1 - \log 2} = 0.43067$

$S_\infty = \log_{10} 2 / (1 - \log_{10} 2) = 0.43068$

204. $(1 - 2x)^7 = (1 + (-2x))^7$
$= 1 + 7(-2x) + \frac{7.6}{2.1}(-2x)^2 + \frac{7.6.5}{3.2.1}(-2x)^3 + \ldots$
$= 1 - 14x + 84x^2 - 280x^3$

205. (i) $S_{12} = \frac{\frac{3}{2}\left(1 - \frac{3}{4}^{12}\right)}{1 - \frac{3}{4}} = 5.81$

(ii) $\frac{3/2}{1 - 3/4} = 6$

206. $\operatorname{cosec}^2 x - 1 = 3\operatorname{cosec} x - 3$
$\operatorname{cosec}^2 x - 3\operatorname{cosec} x + 2 = 0$
$(\operatorname{cosec} x - 1)(\operatorname{cosec} x + 1) = 0$
$\operatorname{cosec} x = 1$ or 2
$\therefore \sin x = 1$ or ½
$x = x = \pi/6, \pi/2, 5\pi/6, 3\pi/2$

207. $(2 - x)^{-1} = \{2(1 - \frac{x}{2})\}^{-1} = 2^{-1}(1 + (-\frac{x}{2}))^{-1}$
$= \frac{1}{2}\{1 + (-1)(-\frac{x}{2}) + \frac{(-1)(-2)}{2.1}(-\frac{x}{2})^2 + \frac{(-1)(-2)(-3)}{3.2.1}(-\frac{x}{2})^3 + .$
$= \frac{1}{2}\{1 + \frac{x}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^3 + ..\} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + ..$
valid for $-1 < -x/2 < 1$ i.e. $-2 < x < 2$

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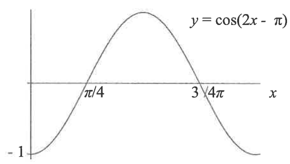
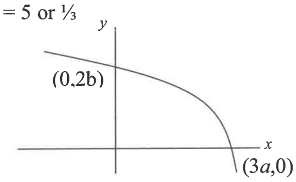
208.



209. (a) $3x - 2 = 4 \Rightarrow x = 2$,
$-(3x - 2) = 4 \Rightarrow x = -2/3$
$\therefore -2/3 < x < 2$
(b) $2x - 3 = x + 2$ and $-(2x - 3) = x + 2$
$x = 5 \qquad -2x + 3 = x + 2$
$x = ⅓$
$\therefore x = 5$ or ⅓

210.



211. (a) $R\cos(x + \alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$
$R\cos\alpha = 1 \qquad R\sin\alpha = \sqrt{3}$
$R = \sqrt{3+1} = \sqrt{4} = 2 \quad \tan\alpha = \sqrt{3}/1 \qquad \alpha = \pi/3$
$\therefore \cos x - \sqrt{3}\sin x = 2\cos(x + \pi/3)$
(b) $\cos(x + \pi/3) = 1$
$(x + \pi/3) = ½$
$x + \pi/3 = \pi/3, 5\pi/3, 7\pi/3, \ldots$
$x = 0, 4\pi/3, 2\pi$

212. $\frac{2\theta\times\theta}{1 - (1 - \frac{4\theta^2}{2})} = \frac{2\theta^2}{\frac{4\theta^2}{2}} = 1$

213. (a)
$\frac{dy}{dx} = -2\times\frac{1}{2}(3x - 1)^{-\frac{3}{2}}\times 3 = -\frac{3}{\sqrt{(3x-1)^3}}$

(b) $\frac{dy}{dx} = \frac{5x\times 2x - (x^2 - 3)5}{25x^2} = \frac{x^2 + 3}{5x^2}$

(c) $\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x} - \frac{1}{2\sqrt{x}} + 8e^{4x}$